ARML Competition 2010

Paul J. Karafiol, Head Writer
Paul Dreyer
Edward Early
Zuming Feng
Benji Fisher
Zachary Franco
Chris Jeuell
Andy Niedermaier
Leo Schneider
Andy Soffer
Eric Wepsic

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1 Individual Problems

Problem 1. Compute the number of positive integers less than 25 that cannot be written as the difference of two squares of integers.

Problem 2. For digits A, B, and C, $(\underline{A}\,\underline{B})^2 + (\underline{A}\,\underline{C})^2 = 1313$. Compute $A + B + C$.

Problem 3. Points P, Q, R, and S lie in the interior of square $ABCD$ such that triangles ABP, BCQ, CDR, and DAS are equilateral. If $AB = 1$, compute the area of quadrilateral $PQRS$.

Problem 4. For real numbers α, B, and C, the zeros of $T(x) = x^3 + x^2 + Bx + C$ are $\sin^2\alpha$, $\cos^2\alpha$, and $-\csc^2\alpha$. Compute $T(5)$.

Problem 5. Let $\mathcal{R}$ denote the circular region bounded by $x^2 + y^2 = 36$. The lines $x = 4$ and $y = 3$ partition $\mathcal{R}$ into four regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, and $\mathcal{R}_4$. $[\mathcal{R}_i]$ denotes the area of region $\mathcal{R}_i$. If $[\mathcal{R}_1] > [\mathcal{R}_2] > [\mathcal{R}_3] > [\mathcal{R}_4]$, compute $[\mathcal{R}_1] - [\mathcal{R}_2] - [\mathcal{R}_3] + [\mathcal{R}_4]$.

Problem 6. Let x be a real number in the interval $[0, 360]$ such that the four expressions $\sin x^\circ$, $\cos x^\circ$, $\tan x^\circ$, $\cot x^\circ$ take on exactly three distinct (finite) real values. Compute the sum of all possible values of x.

Problem 7. Let $a_1, a_2, a_3, \ldots$ be an arithmetic sequence, and let $b_1, b_2, b_3, \ldots$ be a geometric sequence. The sequence $c_1, c_2, c_3, \ldots$ has $c_n = a_n + b_n$ for each positive integer n. If $c_1 = 1, c_2 = 4, c_3 = 15$, and $c_4 = 2$, compute c_5.

Problem 8. In square $ABCD$ with diagonal 1, E is on $\overline{AB}$ and F is on $\overline{BC}$ with $m\angle BCE = m\angle BAF = 30^\circ$. If $\overline{CE}$ and $\overline{AF}$ intersect at G, compute the distance between the incenters of triangles AGE and CGF.

Problem 9. Let a, b, m, n be positive integers with $am = bn = 120$ and $a \neq b$. In the coordinate plane, let $A = (a, m)$, $B = (b, n)$, and $O = (0, 0)$. If X is a point in the plane such that $AOBX$ is a parallelogram, compute the minimum area of $AOBX$.

Problem 10. Let $\mathcal{S}$ be the set of integers from 0 to 9999 inclusive whose base-2 and base-5 representations end in the same four digits. (Leading zeros are allowed, so $1 = 0001_2 = 0001_5$ is one such number.) Compute the remainder when the sum of the elements of $\mathcal{S}$ is divided by 10,000.

2 Individual Answers

Answer 1. 6

Answer 2. 13

Answer 3. $2 - \sqrt{3}$

Answer 4. $\frac{567}{4}$ or equivalent (141.75 or $141\frac{3}{4}$)

Answer 5. 48

Answer 6. 990

Answer 7. 61

Answer 8. $4 - 2\sqrt{3}$

Answer 9. 44

Answer 10. 6248

3 Individual Solutions

Problem 1. Compute the number of positive integers less than 25 that cannot be written as the difference of two squares of integers.

Solution 1. Suppose $n = a^2 - b^2 = (a+b)(a-b)$, where a and b are integers. Because $a+b$ and $a-b$ differ by an even number, they have the same parity. Thus n must be expressible as the product of two even integers or two odd integers. This condition is sufficient for n to be a difference of squares, because if n is odd, then $n = (k+1)^2 - k^2 = (2k+1)\cdot 1$ for some integer k, and if n is a multiple of 4, then $n = (k+1)^2 - (k-1)^2 = 2k \cdot 2$ for some integer k. Therefore any integer of the form $4k+2$ for integral k cannot be expressed as the difference of two squares of integers, hence the desired integers in the given range are $2, 6, 10, 14, 18,$ and 22, for a total of **6** values.

Alternate Solution: Suppose that an integer n can be expressed as the difference of squares of two integers, and let the squares be a^2 and $(a+b)^2$, with $a, b \geq 0$. Then

$$\begin{aligned} n = (a+b)^2 - a^2 &= 2ab + b^2 \\ &= 2a + 1 \quad (b=1) \\ &= 4a + 4 \quad (b=2) \\ &= 6a + 9 \quad (b=3) \\ &= 8a + 16 \quad (b=4) \\ &= 10a + 25 \quad (b=5). \end{aligned}$$

Setting $b = 1$ generates all odd integers. If $b = 3$ or $b = 5$, then the values of n are still odd, hence are already accounted for. If $b = 2$, then the values of $4a + 4 = 4(a+1)$ yield all multiples of 4; $b = 8$ yields multiples of 8 (hence are already accounted for). The remaining integers are even numbers that are not multiples of 4: 2, 6, 10, 14, 18, 22, for a total of **6** such numbers.

Problem 2. For digits $A, B,$ and C, $(\underline{A}\,\underline{B})^2 + (\underline{A}\,\underline{C})^2 = 1313$. Compute $A + B + C$.

Solution 2. Because $10A \leq \underline{A}\,\underline{B} < 10(A+1)$, $200A^2 < (\underline{A}\,\underline{B})^2 + (\underline{A}\,\underline{C})^2 < 200(A+1)^2$. So $200A^2 < 1313 < 200(A+1)^2$, and $A = 2$. Note that B and C must have opposite parity, so without loss of generality, assume that B is even. Consider the numbers modulo 10: for any integer n, $n^2 \equiv 0, 1, 4, 5, 6,$ or $9 \bmod 10$. The only combination whose sum is congruent to $3 \bmod 10$ is $4 + 9$. So $B = 2$ or 8 and $C = 3$ or 7. Checking cases shows that $28^2 + 23^2 = 1313$, so $B = 8, C = 3$, and $A + B + C = \mathbf{13}$.

Alternate Solution: Rewrite $1313 = 13 \cdot 101 = (3^2 + 2^2)(10^2 + 1^2)$. The two-square identity states:

$$\begin{aligned} (a^2 + b^2)(x^2 + y^2) &= (ax + by)^2 + (ay - bx)^2 \\ &= (ay + bx)^2 + (ax - by)^2. \end{aligned}$$

Therefore

$$\begin{aligned} 1313 = (30 + 2)^2 + (3 - 20)^2 &= 32^2 + 17^2 \\ &= (3 + 20)^2 + (30 - 2)^2 = 23^2 + 28^2. \end{aligned}$$

Hence $A = 2, B = 3, C = 8$, and $A + B + C = \mathbf{13}$.

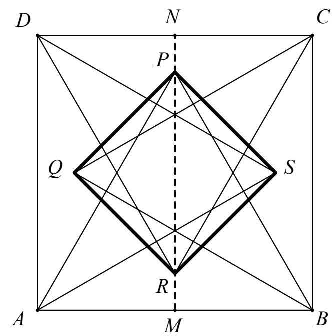
Note: Factoring 1313 into the product of prime Gaussian integers

$$1313 = (3 + i)(3 - i)(10 + i)(10 - i)$$

helps show that this solution is unique. Because factorization in the Gaussian integers is unique (up to factors of i), there is only one way to write 1313 in the form $(a^2+b^2)(x^2+y^2)$. Therefore the values a, b, x, y given in the two-square identity above are uniquely determined (up to permutations).

Problem 3. Points P, Q, R, and S lie in the interior of square $ABCD$ such that triangles ABP, BCQ, CDR, and DAS are equilateral. If $AB = 1$, compute the area of quadrilateral $PQRS$.

Solution 3. $PQRS$ is a square with diagonal $\overline{RP}$. Extend $\overline{RP}$ to intersect $\overline{AB}$ and $\overline{CD}$ at M and N respectively, as shown in the diagram below.



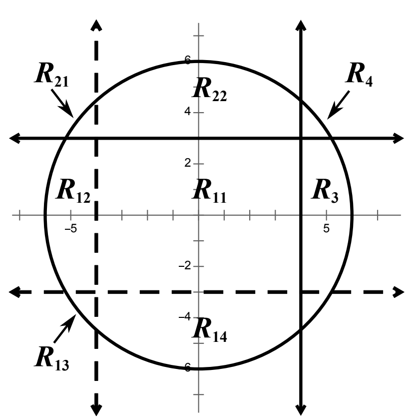
Then $\overline{MP}$ is an altitude of $\triangle ABP$ and $\overline{RN}$ is an altitude of $\triangle CDR$. Adding lengths, $MP + RN = MR + 2RP + PN = 1 + RP$, so $RP = \sqrt{3} - 1$. Therefore $[PQRS] = \frac{1}{2}(RP)^2 = \mathbf{2 - \sqrt{3}}$.

Problem 4. For real numbers α, B, and C, the zeros of $T(x) = x^3 + x^2 + Bx + C$ are $\sin^2\alpha$, $\cos^2\alpha$, and $-\csc^2\alpha$. Compute $T(5)$.

Solution 4. Use the sum of the roots formula to obtain $\sin^2\alpha + \cos^2\alpha + -\csc^2\alpha = -1$, so $\csc^2\alpha = 2$, and $\sin^2\alpha = \frac{1}{2}$. Therefore $\cos^2\alpha = \frac{1}{2}$. $T(x)$ has leading coefficient 1, so by the factor theorem, $T(x) = \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)(x+2)$. Then $T(5) = \left(5 - \frac{1}{2}\right)\left(5 - \frac{1}{2}\right)(5+2) = \mathbf{\frac{567}{4}}$.

Problem 5. Let $\mathcal{R}$ denote the circular region bounded by $x^2 + y^2 = 36$. The lines $x = 4$ and $y = 3$ partition $\mathcal{R}$ into four regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, and $\mathcal{R}_4$. $[\mathcal{R}_i]$ denotes the area of region $\mathcal{R}_i$. If $[\mathcal{R}_1] > [\mathcal{R}_2] > [\mathcal{R}_3] > [\mathcal{R}_4]$, compute $[\mathcal{R}_1] - [\mathcal{R}_2] - [\mathcal{R}_3] + [\mathcal{R}_4]$.

Solution 5. Draw the lines $x = -4$ and $y = -3$, creating regions $\mathcal{R}_{21}, \mathcal{R}_{22}, \mathcal{R}_{11}, \mathcal{R}_{12}, \mathcal{R}_{13}, \mathcal{R}_{14}$ as shown below.



Then $[\mathcal{R}_{21}] = [\mathcal{R}_4] = [\mathcal{R}_{13}]$, $[\mathcal{R}_{22}] = [\mathcal{R}_{14}]$, and $[\mathcal{R}_3] = [\mathcal{R}_{12}] + [\mathcal{R}_{13}]$. Therefore

$$\begin{aligned}
[\mathcal{R}_1] - [\mathcal{R}_2] - [\mathcal{R}_3] + [\mathcal{R}_4] &= ([\mathcal{R}_1] - [\mathcal{R}_2]) - ([\mathcal{R}_3] - [\mathcal{R}_4]) \\
&= ([\mathcal{R}_1] - [\mathcal{R}_{13}] - [\mathcal{R}_{14}]) - ([\mathcal{R}_{12}] + [\mathcal{R}_{13}] - [\mathcal{R}_{21}]) \\
&= ([\mathcal{R}_{11}] + [\mathcal{R}_{12}]) - [\mathcal{R}_{12}] \\
&= [\mathcal{R}_{11}].
\end{aligned}$$

This last region is simply a rectangle of height 6 and width 8, so its area is **48**.

Problem 6. Let x be a real number in the interval $[0, 360]$ such that the four expressions $\sin x^\circ$, $\cos x^\circ$, $\tan x^\circ$, $\cot x^\circ$ take on exactly three distinct (finite) real values. Compute the sum of all possible values of x.

Solution 6. If the four expressions take on three different values, exactly two of the expressions must have equal values. There are $\binom{4}{2} = 6$ cases to consider:

Case 1 $\sin x^\circ = \cos x^\circ$: Then $\tan x^\circ = \cot x^\circ = 1$, violating the condition that there be three distinct values.

Case 2 $\sin x^\circ = \tan x^\circ$: Because $\tan x^\circ = \frac{\sin x^\circ}{\cos x^\circ}$, either $\cos x^\circ = 1$ or $\sin x^\circ = 0$. However, in both of these cases, $\cot x^\circ$ is undefined, so it does not have a real value.

Case 3 $\sin x^\circ = \cot x^\circ$: Then $\sin x^\circ = \frac{\cos x^\circ}{\sin x^\circ}$, and so $\sin^2 x^\circ = \cos x^\circ$. Rewrite using the Pythagorean identity to obtain $\cos^2 x^\circ + \cos x^\circ - 1 = 0$, so $\cos x^\circ = \frac{-1+\sqrt{5}}{2}$ (the other root is outside the range of cos). Because $\cos x^\circ > 0$, this equation has two solutions in $[0, 360]$: an angle x_0° in the first quadrant and the angle $(360 - x_0)^\circ$ in the fourth quadrant. The sum of these two values is 360.

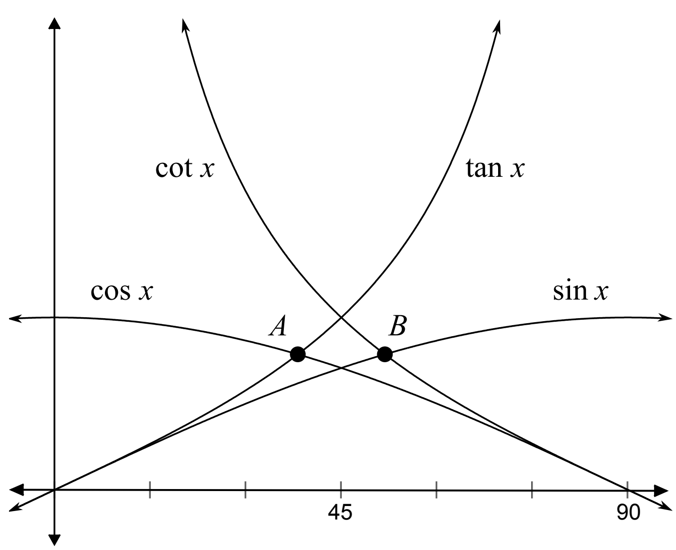
Case 4 $\cos x^\circ = \tan x^\circ$: Use similar logic as in the previous case to obtain the equation $\sin^2 x^\circ + \sin x^\circ - 1 = 0$, so now $\sin x^\circ = \frac{-1+\sqrt{5}}{2}$. Because $\sin x^\circ > 0$, this equation has two solutions, one an angle x_0° in the first quadrant, and the other its supplement $(180 - x_0)^\circ$ in the second quadrant. The sum of these two values is 180.

Case 5 $\cos x^\circ = \cot x^\circ$: In this case, $\tan x^\circ$ is undefined for reasons analogous to those in Case 2.

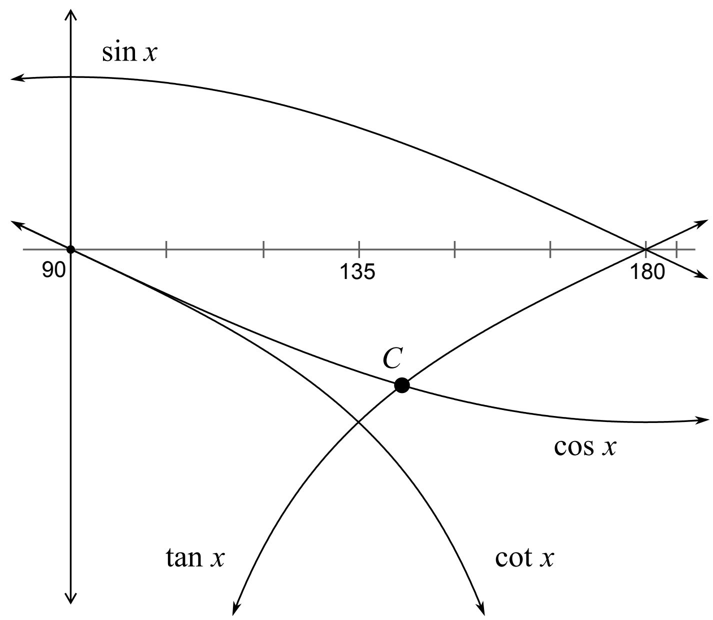
Case 6 $\tan x^\circ = \cot x^\circ$: Thus $\tan^2 x^\circ = 1$, hence $\tan x^\circ = \pm 1$. If $\tan x^\circ = 1$, then $\sin x^\circ = \cos x^\circ$, which yields only two distinct values. So $\tan x^\circ = -1$, which occurs at $x = 135$ and $x = 315$. The sum of these values is 450.

The answer is $360 + 180 + 450 =$ **990**.

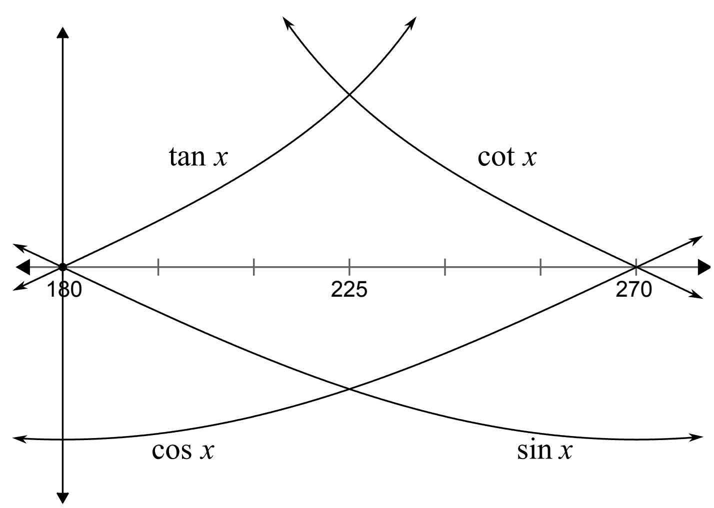
Alternate Solution: Consider the graphs of all four functions; notice first that $0, 90, 180, 270$ are not solutions because either $\tan x^\circ$ or $\cot x^\circ$ is undefined at each value.



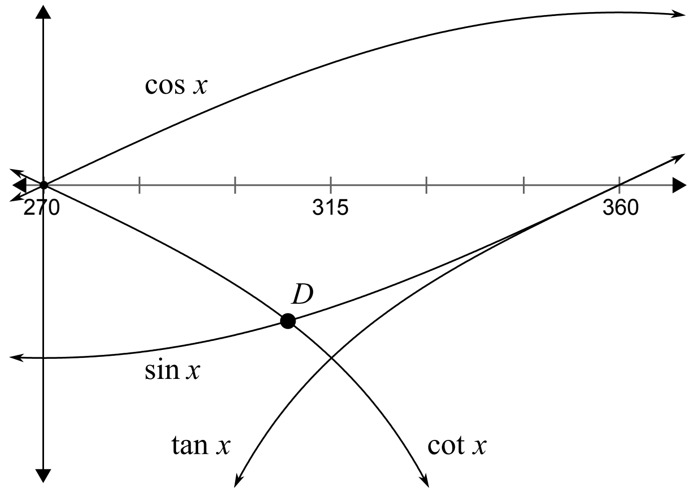
Start in the first quadrant. Let x_1 and x_2 be the values of x such that $\cos x^\circ = \tan x^\circ$ and $\sin x^\circ = \cot^\circ$, respectively, labeled A and B in the diagram. Because $\cos x^\circ = \sin(90 - x)^\circ$ and $\cot x^\circ = \tan(90 - x)^\circ$, $x_1 + x_2 = 90$. One can also see that the graphs of $y = \cot x^\circ$ and $y = \tan x^\circ$ cross at $x = 45$, but so do the graphs of $y = \sin x^\circ$ and $y = \cos x^\circ$. So at $x = 45$, there are only two distinct values, not three.



In the second quadrant, $\tan x^\circ = \cot x^\circ$ when $x = 135$. Also, because $\tan x^\circ$ increases from $-\infty$ to 0 while $\cos x^\circ$ decreases from 0 to -1, there exists a number x_3 such that $\tan x_3^\circ = \cos x_3^\circ$ (marked point C in the diagram above).



In the third quadrant, $\tan x^\circ$ and $\cot x^\circ$ are positive, while $\sin x^\circ$ and $\cos x^\circ$ are negative; the only place where graphs cross is at $x = 225$, but this value is not a solution because the four trigonometric functions have only two distinct values.



In the fourth quadrant, $\tan x^\circ = \cot x^\circ = -1$ when $x = 315$. Because $\sin x^\circ$ is increasing from -1 to 0 while $\cot x^\circ$ is decreasing from 0 to $-\infty$, there exists a number x_4 such that $\sin x_4^\circ = \cot x_4^\circ$ (marked D in the diagram above). Because $\cos x^\circ = \sin(90 - x)^\circ = \sin(450 - x)^\circ$ and $\cot x^\circ = \tan(90 - x)^\circ = \tan(450 - x)^\circ$, the values x_3 and x_4 are symmetrical around $x = 225$, that is, $x_3 + x_4 = 450$.

The sum is $(x_1 + x_2) + (135 + 315) + (x_3 + x_4) = 90 + 450 + 450 = \mathbf{990}$.

Problem 7. Let $a_1, a_2, a_3, \ldots$ be an arithmetic sequence, and let $b_1, b_2, b_3, \ldots$ be a geometric sequence. The sequence $c_1, c_2, c_3, \ldots$ has $c_n = a_n + b_n$ for each positive integer n. If $c_1 = 1, c_2 = 4, c_3 = 15$, and $c_4 = 2$, compute c_5.

Solution 7. Let $a_2 - a_1 = d$ and $\frac{b_2}{b_1} = r$. Using $a = a_1$ and $b = b_1$, write the system of equations:

$$\begin{aligned} a + b &= 1 \\ (a + d) + br &= 4 \\ (a + 2d) + br^2 &= 15 \\ (a + 3d) + br^3 &= 2. \end{aligned}$$

Subtract the first equation from the second, the second from the third, and the third from the fourth to obtain three equations:

$$\begin{aligned} d + b(r - 1) &= 3 \\ d + b(r^2 - r) &= 11 \\ d + b(r^3 - r^2) &= -13. \end{aligned}$$

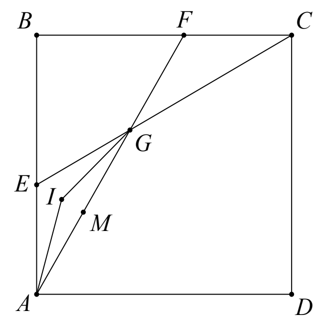
Notice that the a terms have canceled. Repeat to find the second differences:

$$\begin{aligned} b(r^2 - 2r + 1) &= 8 \\ b(r^3 - 2r^2 + r) &= -24. \end{aligned}$$

Now divide the second equation by the first to obtain $r = -3$. Substituting back into either of these two last equations yields $b = \frac{1}{2}$. Continuing in the same vein yields $d = 5$ and $a = \frac{1}{2}$. Then $a_5 = \frac{41}{2}$ and $b_5 = \frac{81}{2}$, so $c_5 = \mathbf{61}$.

Problem 8. In square $ABCD$ with diagonal 1, E is on $\overline{AB}$ and F is on $\overline{BC}$ with $\text{m}\angle BCE = \text{m}\angle BAF = 30^\circ$. If $\overline{CE}$ and $\overline{AF}$ intersect at G, compute the distance between the incenters of triangles AGE and CGF.

Solution 8. Let M be the midpoint of $\overline{AG}$, and I the incenter of $\triangle AGE$ as shown below.



Because $\frac{AB}{AC} = \sin 45°$ and $\frac{EB}{AB} = \frac{EB}{BC} = \tan 30°$,

$$\begin{aligned} AE &= AB - EB = AB(1 - \tan 30°) \\ &= \sin 45°(1 - \tan 30°) \\ &= \frac{\sin 45° \cos 30° - \cos 45° \sin 30°}{\cos 30°} \\ &= \frac{\sin(45° - 30°)}{\cos 30°} \\ &= \frac{\sin 15°}{\cos 30°}. \end{aligned}$$

Note that $\frac{AM}{AE} = \cos 30°$ and $\frac{AM}{AI} = \cos 15°$. Therefore

$$\begin{aligned} \frac{AI}{AE} &= \frac{\cos 30°}{\cos 15°} \\ &= \frac{\sin 60°}{\cos 15°} \\ &= \frac{2 \sin 30° \cos 30°}{\cos 15°} \\ &= \frac{2(2 \sin 15° \cos 15°) \cos 30°}{\cos 15°} \\ &= 4 \sin 15° \cos 30°. \end{aligned}$$

Thus $AI = (4 \sin 15° \cos 30°)(\frac{\sin 15°}{\cos 30°}) = 4 \sin^2 15° = 4(\frac{1-\cos 30°}{2}) = 2 - \sqrt{3}$. Finally, the desired distance is $2IG = 2AI = \mathbf{4 - 2\sqrt{3}}$.

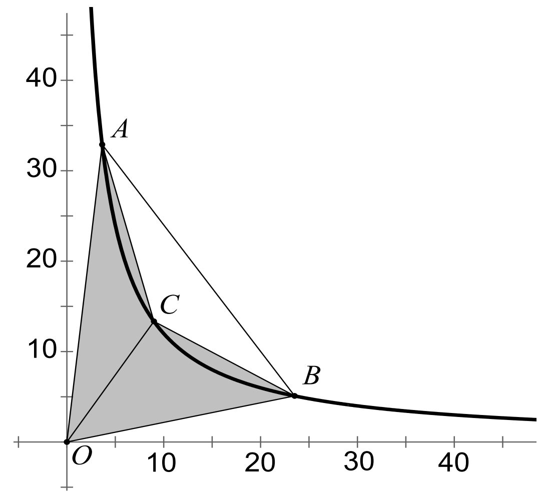
Problem 9. Let a, b, m, n be positive integers with $am = bn = 120$ and $a \neq b$. In the coordinate plane, let $A = (a, m)$, $B = (b, n)$, and $O = (0, 0)$. If X is a point in the plane such that $AOBX$ is a parallelogram, compute the minimum area of $AOBX$.

Solution 9. The area of parallelogram $AOBX$ is given by the absolute value of the cross product $|\langle a, m\rangle \times \langle b, n\rangle| = |an - mb|$. Because $m = \frac{120}{a}$ and $n = \frac{120}{b}$, the desired area of $AOBX$ equals $120 \left|\frac{a}{b} - \frac{b}{a}\right|$. Note that the function $f(x) = x - 1/x$ is monotone increasing for $x > 1$. (Proof: if $x_1 > x_2 > 0$, then $f(x_1) - f(x_2) = (x_1 - x_2) + \frac{x_1 - x_2}{x_1 x_2}$, where both terms are positive because $x_1 x_2 > 0$.) So the minimum value of $[AOBX]$ is attained when $\frac{a}{b}$ is as close as possible to 1, that is, when a and b are consecutive divisors of 120. By symmetry, consider only $a < b$; notice too that because $\frac{120/a}{120/b} = \frac{b}{a}$, only values with $b \leq \sqrt{120}$ need be considered. These observations can be used to generate the table below:

a, m	$1, 120$	$2, 60$	$3, 40$	$4, 30$	$5, 24$	$6, 20$	$8, 15$	$10, 12$
b, n	$2, 60$	$3, 40$	$4, 30$	$5, 24$	$6, 20$	$8, 15$	$10, 12$	$12, 10$
$[AOBX]$	180	100	70	54	44	70	54	44

The smallest value is **44**, achieved using $(5, 24)$ and $(6, 20)$, or using $(10, 12)$ and $(12, 10)$.

Note: The fact that a and b must be consecutive divisors of 120 can also be established by the following geometric argument. Notice that $[AOBX] = 2[AOB]$. Suppose C is a point on the hyperbola $y = 120/x$ between A and B, as shown in the diagram below.



Because the hyperbola is concave up, $[OAC] + [OCB] < [OAB]$, so in particular, $[OAC] < [OAB]$. Thus, if $[OAB]$ is minimal, there can be no point C with integer coordinates between A and B on the hyperbola.

Problem 10. Let $\mathcal{S}$ be the set of integers from 0 to 9999 inclusive whose base-2 and base-5 representations end in the same four digits. (Leading zeros are allowed, so $1 = 0001_2 = 0001_5$ is one such number.) Compute the remainder when the sum of the elements of $\mathcal{S}$ is divided by 10,000.

Solution 10. The remainders of an integer N modulo $2^4 = 16$ and $5^4 = 625$ uniquely determine its remainder modulo 10000. There are only 16 strings of four 0's and 1's. In addition, because 16 and 625 are relatively prime, it will be shown below that for each such string s, there exists exactly one integer x_s in the range $0 \leq x_s < 10000$ such that the base-2 and base-5 representations of x_s end in the digits of s (e.g., x_{1001} is the unique positive integer less than 10000 such that x's base-5 representation and base-2 representation both end in 1001).

Here is a proof of the preceding claim: Let $p(s)$ be the number whose digits in base 5 are the string s, and $b(s)$ be the number whose digits in base 2 are the string s. Then the system $x \equiv p(s) \bmod 625$ and $x \equiv b(s) \bmod 16$ can be rewritten as $x = p(s) + 625m$ and $x = b(s) + 16n$ for integers m and n. These reduce to the Diophantine equation $16n - 625m = p(s) - b(s)$, which has solutions m, n in $\mathbb{Z}$, with at least one of $m, n \geq 0$. Assuming without loss of generality that $m > 0$ yields $x = p(s) + 625m \geq 0$. To show that there exists an $x_s < 10000$ and that it is unique, observe that the general form of the solution is $m' = m - 16t, n' = n + 625t$. Thus if $p(s) + 625m > 10000$, an appropriate t can be found by writing $0 \leq p(s) + 625(m - 16t) < 10000$, which yields $p(s) + 625m - 10000 < 10000t \leq p(s) + 625m$. Because there are exactly 10000 integers in that interval, exactly one of them is divisible by 10000, so there is exactly one value of t satisfying $0 \leq p(s) + 625(m - 16t) < 10000$, and set $x_s = 625(m - 16t)$.

Therefore there will be 16 integers whose base-2 and base-5 representations end in the same four digits, possibly with leading 0's as in the example. Let $X = x_{0000} + \cdots + x_{1111}$. Then X is congruent modulo 16 to

$0000_2 + \cdots + 1111_2 = 8 \cdot (1111_2) = 8 \cdot 15 \equiv 8$. Similarly, X is congruent modulo 625 to $0000_5 + \cdots + 1111_5 = 8 \cdot 1111_5 = 2 \cdot 4444_5 \equiv 2 \cdot (-1) = -2$.

So X must be 8 (mod 16) and -2 (mod 625). Noticing that $625 \equiv 1 \pmod{16}$, conclude that the answer is $-2 + 10 \cdot 625 = \mathbf{6248}$.

4 Team Problems

Problem 1. Compute all ordered pairs of real numbers (x, y) that satisfy both of the equations:

$$x^2 + y^2 = 6y - 4x + 12 \qquad \text{and} \qquad 4y = x^2 + 4x + 12.$$

Problem 2. Define $\log^*(n)$ to be the smallest number of times the log function must be iteratively applied to n to get a result less than or equal to 1. For example, $\log^*(1000) = 2$ since $\log 1000 = 3$ and $\log(\log 1000) = \log 3 = 0.477\ldots \leq 1$. Let a be the smallest integer such that $\log^*(a) = 3$. Compute the number of zeros in the base 10 representation of a.

Problem 3. An integer N is worth 1 point for each pair of digits it contains that forms a prime in its original order. For example, 6733 is worth 3 points (for 67, 73, and 73 again), and 20304 is worth 2 points (for 23 and 03). Compute the smallest positive integer that is worth exactly 11 points. [Note: Leading zeros are not allowed in the original integer.]

Problem 4. The six sides of convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. Compute the number of colorings such that every triangle $A_iA_jA_k$ has at least one red side.

Problem 5. Compute the smallest positive integer n such that n^n has at least 1,000,000 positive divisors.

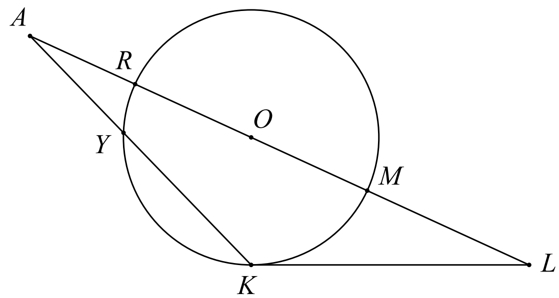
Problem 6. Given an arbitrary finite sequence of letters (represented as a word), a *subsequence* is a sequence of one or more letters that appear in the same order as in the original sequence. For example, N, CT, OTT, and $CONTEST$ are subsequences of the word $CONTEST$, but NOT, $ONSET$, and $TESS$ are not. Assuming the standard English alphabet $\{A, B, \ldots, Z\}$, compute the number of distinct four-letter "words" for which EE is a subsequence.

Problem 7. Six solid regular tetrahedra are placed on a flat surface so that their bases form a regular hexagon $\mathcal{H}$ with side length 1, and so that the vertices not lying in the plane of $\mathcal{H}$ (the "top" vertices) are themselves coplanar. A spherical ball of radius r is placed so that its center is directly above the center of the hexagon. The sphere rests on the tetrahedra so that it is tangent to one edge from each tetrahedron. If the ball's center is coplanar with the top vertices of the tetrahedra, compute r.

Problem 8. Derek starts at the point $(0, 0)$, facing the point $(0, 1)$, and he wants to get to the point $(1, 1)$. He takes unit steps parallel to the coordinate axes. A move consists of either a step forward, or a $90°$ right (clockwise) turn followed by a step forward, so that his path does not contain any left turns. His path is restricted to the square region defined by $0 \leq x \leq 17$ and $0 \leq y \leq 17$. Compute the number of ways he can get to $(1, 1)$ without returning to any previously visited point.

Problem 9. The equations $x^3 + Ax + 10 = 0$ and $x^3 + Bx^2 + 50 = 0$ have two roots in common. Compute the product of these common roots.

Problem 10. Points A and L lie outside circle ω, whose center is O, and $\overline{AL}$ contains diameter $\overline{RM}$, as shown below. Circle ω is tangent to $\overline{LK}$ at K. Also, $\overline{AK}$ intersects ω at Y, which is between A and K. If $KL = 3$, $ML = 2$, and $\text{m}\angle AKL - \text{m}\angle YMK = 90°$, compute $[AKM]$ (i.e., the area of $\triangle AKM$).


A
R
O
Y
M
L
K

5 Team Answers

Answer 1. $(-6, 6)$ and $(2, 6)$ [must have both answers, in either order]

Answer 2. 9

Answer 3. 100337

Answer 4. 392

Answer 5. 84

Answer 6. 3851

Answer 7. $\frac{\sqrt{2}}{3}$

Answer 8. 529

Answer 9. $5\sqrt[3]{4}$

Answer 10. $\frac{375}{182}$

6 Team Solutions

Problem 1. Compute all ordered pairs of real numbers (x, y) that satisfy both of the equations:

$$x^2 + y^2 = 6y - 4x + 12 \qquad \text{and} \qquad 4y = x^2 + 4x + 12.$$

Solution 1. Rearrange the terms in the first equation to yield $x^2 + 4x + 12 = 6y - y^2 + 24$, so that the two equations together yield $4y = 6y - y^2 + 24$, or $y^2 - 2y - 24 = 0$, from which $y = 6$ or $y = -4$. If $y = 6$, then $x^2 + 4x + 12 = 24$, from which $x = -6$ or $x = 2$. If $y = -4$, then $x^2 + 4x + 12 = -16$, which has no real solutions because $x^2 + 4x + 12 = (x + 2)^2 + 8 \geq 8$ for all real x. So there are two ordered pairs satisfying the system, namely $(-6, 6)$ and $(2, 6)$.

Problem 2. Define $\log^*(n)$ to be the smallest number of times the log function must be iteratively applied to n to get a result less than or equal to 1. For example, $\log^*(1000) = 2$ since $\log 1000 = 3$ and $\log(\log 1000) = \log 3 = 0.477\ldots \leq 1$. Let a be the smallest integer such that $\log^*(a) = 3$. Compute the number of zeros in the base 10 representation of a.

Solution 2. If $\log^*(a) = 3$, then $\log(\log(\log(a))) \leq 1$ and $\log(\log(a)) > 1$. If $\log(\log(a)) > 1$, then $\log(a) > 10$ and $a > 10^{10}$. Because the problem asks for the smallest such a that is an integer, choose $a = 10^{10} + 1 =$ 10,000,000,001, which has **9** zeros.

Problem 3. An integer N is worth 1 point for each pair of digits it contains that forms a prime in its original order. For example, 6733 is worth 3 points (for 67, 73, and 73 again), and 20304 is worth 2 points (for 23 and 03). Compute the smallest positive integer that is worth exactly 11 points. [Note: Leading zeros are <u>not</u> allowed in the original integer.]

Solution 3. If a number N has k base 10 digits, then its maximum point value is $(k-1) + (k-2) + \cdots + 1 = \frac{1}{2}(k-1)(k)$. So if $k \leq 5$, the number N is worth at most 10 points. Therefore the desired number has at least six digits. If $100{,}000 < N < 101{,}000$, then N is of the form $100\underline{A}\,\underline{B}\,\underline{C}$, which could yield 12 possible primes, namely $1\underline{A}$, $1\underline{B}$, $1\underline{C}$, $0\underline{A}$ (twice), $0\underline{B}$ (twice), $0\underline{C}$ (twice), $\underline{A}\,\underline{B}$, $\underline{A}\,\underline{C}$, $\underline{B}\,\underline{C}$. So search for N of the form $100\underline{A}\,\underline{B}\,\underline{C}$, starting with lowest values first. Notice that if any of A, B, or C is not a prime, at least two points are lost, so all three numbers must be prime. Proceed by cases:

First consider the case $A = 2$. Then $1\underline{A}$ is composite, so all of $1\underline{B}$, $1\underline{C}$, $\underline{A}\,\underline{B}$, $\underline{A}\,\underline{C}$, $\underline{B}\,\underline{C}$ must be prime. Considering now the values of $1\underline{B}$ and $1\underline{C}$, both B and C must be in the set $\{3, 7\}$. Because 27 is composite, $B = C = 3$, but then $\underline{B}\,\underline{C} = 33$ is composite. So A cannot equal 2.

If $A = 3$, then $B \neq 2$ because both 12 and 32 are composite. If $B = 3$, $1\underline{B}$ is prime but $\underline{A}\,\underline{B} = 33$ is composite, so all of C, $1\underline{C}$, and $3\underline{C}$ must be prime. These conditions are satisfied by $C = 7$ and no other value. So $A = B = 3$ and $C = 7$, yielding $N =$ **100337**.

Problem 4. The six sides of convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. Compute the number of colorings such that every triangle $A_iA_jA_k$ has at least one red side.

Solution 4. Only two triangles have no sides that are sides of the original hexagon: $A_1A_3A_5$ and $A_2A_4A_6$. For each of these triangles, there are $2^3 - 1 = 7$ colorings in which at least one side is red, for a total of $7 \cdot 7 = 49$ colorings of those six diagonals. The colorings of the three central diagonals $\overline{A_1A_4}, \overline{A_2A_5}, \overline{A_3A_6}$ are irrelevant because the only triangles they can form include sides of the original hexagon, so they can be colored in $2^3 = 8$ ways, for a total of $8 \cdot 49 =$ **392** colorings.

Problem 5. Compute the smallest positive integer n such that n^n has at least 1,000,000 positive divisors.

Solution 5. Let k denote the number of distinct prime divisors of n, so that $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, $a_i > 0$. Then if $d(x)$ denotes the number of positive divisors of x,

$$d\left(n^n\right) = \left(a_1 n + 1\right)\left(a_2 n + 1\right) \cdots \left(a_k n + 1\right) \geq (n+1)^k. \qquad (*)$$

Note that if $n \geq 99$ and $k \geq 3$, then $d(n^n) \geq 100^3 = 10^6$, so $102 = 2 \cdot 3 \cdot 17$ is an upper bound for the solution. Look for values less than 99, using two observations: (1) all $a_i \leq 6$ (because $p^7 > 99$ for all primes); and (2) $k \leq 3$ (because $2 \cdot 3 \cdot 5 \cdot 7 > 99$). These two facts rule out the cases $k = 1$ (because $(*)$ yields $d \leq (6n+1)^1 < 601$) and $k = 2$ (because $d(n^n) \leq (6n+1)^2 < 601^2$).

So $k = 3$. Note that if $a_1 = a_2 = a_3 = 1$, then from $(*)$, $d(n^n) = (n+1)^3 < 10^6$. So consider only $n < 99$ with exactly three prime divisors, and for which not all exponents are 1. The only candidates are 60, 84, and 90; of these, $n = 84$ is the smallest one that works:

$$\begin{aligned} d(60^{60}) &= d(2^{120} \cdot 3^{60} \cdot 5^{60}) = 121 \cdot 61 \cdot 61 < 125 \cdot 80 \cdot 80 = 800{,}000; \\ d(84^{84}) &= d(2^{168} \cdot 3^{84} \cdot 7^{84}) = 169 \cdot 85 \cdot 85 > 160 \cdot 80 \cdot 80 = 1{,}024{,}000. \end{aligned}$$

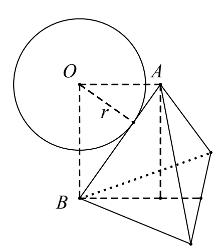
Therefore $n = \mathbf{84}$ is the least positive integer n such that $d(n^n) > 1{,}000{,}000$.

Problem 6. Given an arbitrary finite sequence of letters (represented as a word), a *subsequence* is a sequence of one or more letters that appear in the same order as in the original sequence. For example, N, CT, OTT, and $CONTEST$ are subsequences of the word $CONTEST$, but NOT, $ONSET$, and $TESS$ are not. Assuming the standard English alphabet $\{A, B, \ldots, Z\}$, compute the number of distinct four-letter "words" for which EE is a subsequence.

Solution 6. Divide into cases according to the number of E's in the word. If there are only two E's, then the word must have two non-E letters, represented by ?'s. There are $\binom{4}{2} = 6$ arrangements of two E's and two ?'s, and each of the ?'s can be any of 25 letters, so there are $6 \cdot 25^2 = 3750$ possible words. If there are three E's, then the word has exactly one non-E letter, and so there are 4 arrangements times 25 choices for the letter, or 100 possible words. There is one word with four E's, hence a total of **3851** words.

Problem 7. Six solid regular tetrahedra are placed on a flat surface so that their bases form a regular hexagon $\mathcal{H}$ with side length 1, and so that the vertices not lying in the plane of $\mathcal{H}$ (the "top" vertices) are themselves coplanar. A spherical ball of radius r is placed so that its center is directly above the center of the hexagon. The sphere rests on the tetrahedra so that it is tangent to one edge from each tetrahedron. If the ball's center is coplanar with the top vertices of the tetrahedra, compute r.

Solution 7. Let O be the center of the sphere, A be the top vertex of one tetrahedron, and B be the center of the hexagon.



Then BO equals the height of the tetrahedron, which is $\frac{\sqrt{6}}{3}$. Because A is directly above the centroid of the bottom face, AO is two-thirds the length of the median of one triangular face, so $AO = \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{3}$. The radius of the sphere is the altitude to hypotenuse $\overline{AB}$ of $\triangle ABO$, so the area of $\triangle ABO$ can be represented in two ways: $[ABO] = \frac{1}{2}AO\cdot BO = \frac{1}{2}AB\cdot r$. Substitute given and computed values to obtain $\frac{1}{2}\left(\frac{\sqrt{3}}{3}\right)\left(\frac{\sqrt{6}}{3}\right) = \frac{1}{2}(1)(r)$, from which $r = \frac{\sqrt{18}}{9} = \mathbf{\frac{\sqrt{2}}{3}}$.

Problem 8. Derek starts at the point $(0, 0)$, facing the point $(0, 1)$, and he wants to get to the point $(1, 1)$. He takes unit steps parallel to the coordinate axes. A move consists of either a step forward, or a $90°$ right (clockwise) turn followed by a step forward, so that his path does not contain any left turns. His path is restricted to the square region defined by $0 \le x \le 17$ and $0 \le y \le 17$. Compute the number of ways he can get to $(1, 1)$ without returning to any previously visited point.

Solution 8. Divide into cases according to the number of right turns Derek makes.

- There is one route involving only one turn: move first to $(0, 1)$ and then to $(1, 1)$.
- If he makes two turns, he could move up to $(0, a)$ then to $(1, a)$ and then down to $(1, 1)$. In order to do this, a must satisfy $1 < a \le 17$, leading to 16 options.
- If Derek makes three turns, his path is entirely determined by the point at which he turns for the second time. If the coordinates of this second turn point are (a, b), then both a and b are between 2 and 17 inclusive, yielding $(17 - 1)^2$ possibilities.
- If Derek makes four turns, his last turn must be from facing in the $-x$-direction to the $+y$-direction. For this to be his last turn, it must occur at $(1, 0)$. Then his next-to-last turn could be at any $(a, 0)$, with $1 < a \le 17$, depending on the location of his second turn as in the previous case. This adds another $(17 - 1)^2$ possibilities.
- It is impossible for Derek to make more than four turns and get to $(1, 1)$ without crossing or overlapping his path.

Summing up the possibilities gives $1 + 16 + 16^2 + 16^2 = \mathbf{529}$ possibilities.

Problem 9. The equations $x^3 + Ax + 10 = 0$ and $x^3 + Bx^2 + 50 = 0$ have two roots in common. Compute the product of these common roots.

Solution 9. Let the roots of the first equation be p, q, r and the roots of the second equation be p, q, s. Then $pqr = -10$ and $pqs = -50$, so $\frac{s}{r} = 5$. Also $p + q + r = 0$ and $p + q + s = -B$, so $r - s = B$. Substituting yields $r - 5r = -4r = B$, so $r = -\frac{B}{4}$ and $s = -\frac{5B}{4}$. From the second given equation, $pq + ps + qs = pq + s(p + q) = 0$, so $pq - \frac{5B}{4}(p + q) = 0$, or $pq = \frac{5B}{4}(p + q)$. Because $p + q + r = 0$, $p + q = -r = \frac{B}{4}$, and so $pq = \frac{5B^2}{16}$. Because $pqr = -10$ and $r = -\frac{B}{4}$, conclude that $pq = \frac{40}{B}$. Thus $\frac{5B^2}{16} = \frac{40}{B}$, so $B^3 = 128$ and $B = 4\sqrt[3]{2}$. Then $pq = \frac{5B^2}{16}$ implies that $pq = \mathbf{5\sqrt[3]{4}}$ (and $r = -\sqrt[3]{2}$).

Second Solution: Let the common roots be p and q. Then the following polynomials (linear combinations of the originals) must also have p and q as common zeros:

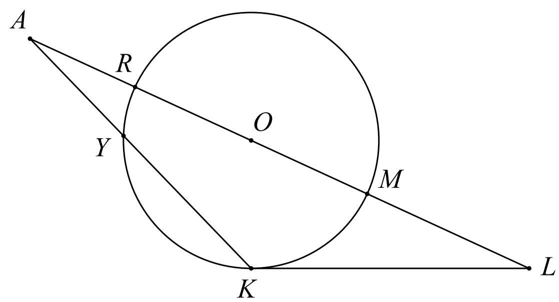
$$\begin{aligned}(x^3 + Bx^2 + 50) - (x^3 + Ax + 10) &= Bx^2 - Ax + 40\\ -(x^3 + Bx^2 + 50) + 5(x^3 + Ax + 10) &= 4x^3 - Bx^2 + 5Ax.\end{aligned}$$

Because $pq \ne 0$, neither p nor q is zero, so the second polynomial has zeros p, q, and 0. Therefore p and q are zeros of $4x^2 - Bx + 5A$. [This result can also be obtained by using the Euclidean Algorithm on the original polynomials.]

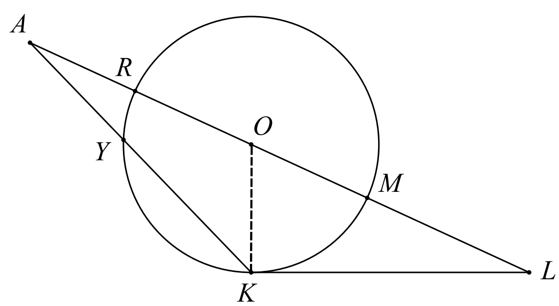
Because the two quadratic equations have the same zeros, their coefficients are proportional: $\frac{4}{B} = \frac{5A}{40} \Rightarrow AB = 32$ and $\frac{4}{B} = \frac{-B}{-A} \Rightarrow 4A = B^2$. Hence $\frac{128}{B} = B^2$ and $B^3 = 128$, so $B = 4\sqrt[3]{2}$. Rewriting the first quadratic as $B\left(x^2 - \frac{A}{B}x + \frac{40}{B}\right)$ shows that the product $pq = \frac{40}{B} = \mathbf{5\sqrt[3]{4}}$.

Third Solution: Using the sum of roots formulas, notice that $pq + ps + qs = p + q + r = 0$. Therefore $0 = pq + ps + qs - (p + q + r)s = pq - rs$, and $pq = rs$. Hence $(pq)^3 = (pqr)(pqs) = (-10)(-50) = 500$, so $pq = \mathbf{5\sqrt[3]{4}}$.

Problem 10. Points A and L lie outside circle ω, whose center is O, and $\overline{AL}$ contains diameter $\overline{RM}$, as shown below. Circle ω is tangent to $\overline{LK}$ at K. Also, $\overline{AK}$ intersects ω at Y, which is between A and K. If $KL = 3$, $ML = 2$, and $m\angle AKL - m\angle YMK = 90°$, compute $[AKM]$ (i.e., the area of $\triangle AKM$).



Solution 10. Notice that $\overline{OK} \perp \overline{KL}$, and let r be the radius of ω.



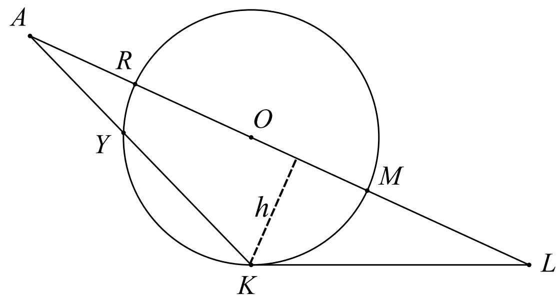
Then consider right triangle OKL. Because $ML = 2$, $OK = r$, and $OL = r+2$, it follows that $r^2+3^2 = (r+2)^2$, from which $r = \frac{5}{4}$.

Because $m\angle YKL = \frac{1}{2}m\widehat{YRK}$ and $m\angle YMK = \frac{1}{2}m\widehat{YK}$, it follows that $m\angle YKL + m\angle YMK = 180°$. By the given condition, $m\angle YKL - m\angle YMK = 90°$. It follows that $m\angle YMK = 45°$ and $m\angle YKL = 135°$. hence $m\widehat{YK} = 90°$. Thus,

$$\overline{YO} \perp \overline{OK} \quad \text{and} \quad \overline{YO} \parallel \overline{KL}. \qquad (*)$$

From here there are several solutions:

First Solution: Compute $[AKM]$ as $\frac{1}{2}$base · height, using base $\overline{AM}$.



Because of ($*$), $\triangle AYO \sim \triangle AKL$. To compute AM, notice that in $\triangle AYO$, $AO = AM - r$, while in $\triangle AKL$, the corresponding side $AL = AM + ML = AM + 2$. Therefore:

$$\begin{aligned} \frac{AO}{AL} &= \frac{YO}{KL} \\ \frac{AM - \frac{5}{4}}{AM + 2} &= \frac{\frac{5}{4}}{3}, \end{aligned}$$

from which $AM = \frac{25}{7}$. Draw the altitude of $\triangle AKM$ from vertex K, and let h be its length. In right triangle OKL, h is the altitude to the hypotenuse, so $\frac{h}{3} = \sin(\angle KLO) = \frac{r}{r+2}$. Hence $h = \frac{15}{13}$. Therefore $[AKM] = \frac{1}{2} \cdot \frac{25}{7} \cdot \frac{15}{13} = \mathbf{\frac{375}{182}}$.

Second Solution: By the Power of the Point Theorem, $LK^2 = LM \cdot LR$, so

$$\begin{aligned} LR &= \frac{9}{2}, \\ RM &= LR - LM = \frac{5}{2}, \\ OL &= r + ML = \frac{13}{4}. \quad (\dagger) \end{aligned}$$

From ($*$), we know that $\triangle AYO \sim \triangle AKL$. hence by ($\dagger$),

$$\frac{AL}{AO} = \frac{AL}{AL - OL} = \frac{KL}{YO} = \frac{3}{\frac{5}{4}} = \frac{12}{5}, \quad \text{thus} \quad AL = \frac{12}{7} \cdot OL = \frac{12}{7} \cdot \frac{13}{4} = \frac{39}{7}.$$

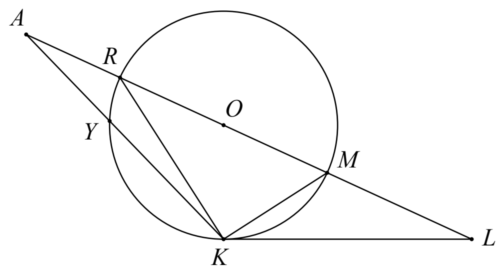
Hence $AM = AL - 2 = \frac{25}{7}$. The ratio between the areas of triangles AKM and RKM is equal to

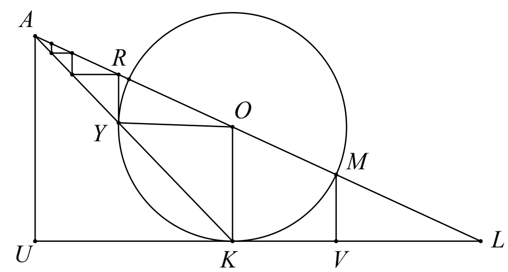
$$\frac{[AKM]}{[RKM]} = \frac{AM}{RM} = \frac{\frac{25}{7}}{\frac{5}{2}} = \frac{10}{7}.$$

Thus $[AKM] = \frac{10}{7} \cdot [RKM]$.

Because $\angle KRL$ and $\angle MKL$ both subtend $\widehat{KM}$, $\triangle KRL \sim \triangle MKL$. Therefore $\frac{KR}{MK} = \frac{LK}{LM} = \frac{3}{2}$. Thus let $KR = 3x$ and $MK = 2x$ for some positive real number x. Because RM is a diameter of ω (see left diagram below), $m\angle RKM = 90°$. Thus triangle RKM is a right triangle with hypotenuse $\overline{RM}$. In particular, $13x^2 = KR^2 + MK^2 = RM^2 = \frac{25}{4}$, so $x^2 = \frac{25}{52}$ and $[RKM] = \frac{RK \cdot KM}{2} = 3x^2$. Therefore

$$[AKM] = \frac{10}{7} \cdot [RKM] = \frac{10}{7} \cdot 3 \cdot \frac{25}{52} = \mathbf{\frac{375}{182}}.$$





Third Solution: Let U and V be the respective feet of the perpendiculars dropped from A and M to $\overleftrightarrow{KL}$. From $(*)$, $\triangle AKL$ can be dissected into two infinite progressions of triangles: one progression of triangles similar to $\triangle OKL$ and the other similar to $\triangle YOK$, as shown in the right diagram above. In both progressions, the corresponding sides of the triangles have common ratio equal to

$$\frac{YO}{KL} = \frac{\frac{5}{4}}{3} = \frac{5}{12}.$$

Thus

$$AU = \frac{5}{4}\left(1 + \frac{5}{12} + \left(\frac{5}{12}\right)^2 + \cdots\right) = \frac{5}{4}\cdot\frac{12}{7} = \frac{15}{7}.$$

Because $\triangle LMV \sim \triangle LOK$, and because $LO = \frac{13}{4}$ by (‡),

$$\frac{MV}{OK} = \frac{LM}{LO}, \quad \text{thus} \quad MV = \frac{OK \cdot LM}{LO} = \frac{\frac{5}{4}\cdot 2}{\frac{13}{4}} = \frac{10}{13}.$$

Finally, note that $[AKM] = [AKL] - [KLM]$. Because $\triangle AKL$ and $\triangle KLM$ share base $\overline{KL}$,

$$[AKM] = \frac{1}{2}\cdot 3\cdot\left(\frac{15}{7} - \frac{10}{13}\right) = \mathbf{\frac{375}{182}}.$$

7 Power Question 2010: Power of Circular Subdivisions

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice-versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

A king strapped for cash is forced to sell off his kingdom $U = \left\{(x, y) : x^2 + y^2 \leq 1\right\}$. He sells the two circular plots C and C' centered at $\left(\pm\frac{1}{2}, 0\right)$ with radius $\frac{1}{2}$. The retained parts of the kingdom form two regions, each bordered by three arcs of circles; in what follows, we will call such regions *curvilinear triangles,* or *c-triangles* (c$\triangle$) for short.

This sad day marks day 0 of a new fiscal era. Unfortunately, these drastic measures are not enough, and so each day thereafter, court geometers mark off the largest possible circle contained in each c-triangle in the remaining property. This circle is tangent to all three arcs of the c-triangle, and will be referred to as the *incircle* of the c-triangle. At the end of the day, all incircles demarcated that day are sold off, and the following day, the remaining c-triangles are partitioned in the same manner.

1. Without using Descartes' Circle Formula (see below):

- **a.** Show that the circles marked off and sold on day 1 are centered at $\left(0, \pm\frac{2}{3}\right)$ with radius $\frac{1}{3}$. [2]
- **b.** Find the combined area of the six remaining curvilinear territories. [2]

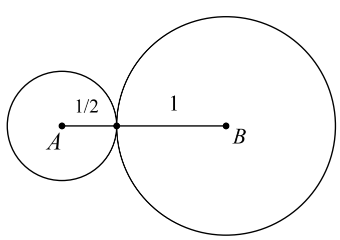
On day 2, the plots bounded by the incircles of the six remaining curvilinear territories are sold.

2a. Determine the number of curvilinear territories remaining at the *end* of day 3. [2]

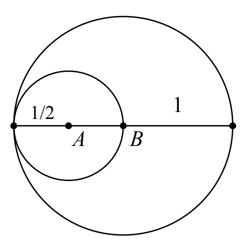
2b. Let X_n be the number of plots sold on day n. Find a formula for X_n in terms of n. [2]

2c. Determine the total number of plots sold up to and including day n. [2]

Some notation: when discussing mutually tangent circles (or arcs), it is convenient to refer to the curvature of a circle rather than its radius. We define *curvature* as follows. Suppose that circle A of radius r_a is externally tangent to circle B of radius r_b. Then the curvatures of the circles are simply the reciprocals of their radii, $\frac{1}{r_a}$ and $\frac{1}{r_b}$. If circle A is internally tangent to circle B, however, as in the right diagram below, the curvature of circle A is still $\frac{1}{r_a}$, while the curvature of circle B is $-\frac{1}{r_b}$, the opposite of the reciprocal of its radius.



Circle A has curvature 2; circle B has curvature 1.



Circle A has curvature 2; circle B has curvature -1.

Using these conventions allows us to express a beautiful theorem of Descartes: when four circles A, B, C, D are pairwise tangent, with respective curvatures a, b, c, d, then

$$(a + b + c + d)^2 = 2\left(a^2 + b^2 + c^2 + d^2\right),$$

where (as before) a is taken to be negative if B, C, D are internally tangent to A, and correspondingly for b,c, or d. This Power Question does not involve the proof of Descartes' Circle Formula, but the formula may be used for all problems below.

3a. Two unit circles and a circle of radius $\frac{2}{3}$ are mutually externally tangent. Compute all possible values of r such that a circle of radius r is tangent to all three circles. [2]

3b. Given three mutually tangent circles with curvatures $a, b, c > 0$, suppose that $(a, b, c, 0)$ does not satisfy Descartes' Circle Formula. Show that there are two distinct values of r such that there is a circle of radius r tangent to the given circles. [3]

3c. Algebraically, it is possible for a quadruple $(a, b, c, 0)$ to satisfy Descartes' Circle Formula, as occurs when $a = b = 1$ and $c = 4$. Find a geometric interpretation for this situation. [2]

4. Let $\phi = \frac{1+\sqrt{5}}{2}$, and let $\rho = \phi + \sqrt{\phi}$.

a. Prove that $\rho^4 = 2\rho^3 + 2\rho^2 + 2\rho - 1$. [2]

b. Show that four pairwise externally tangent circles with nonequal radii in geometric progression must have common ratio ρ. [2]

As shown in problem 3, given A, B, C, D as above with $s = a + b + c + d$, there is a second circle A' with curvature a' also tangent to B, C, and D. We can describe A and A' as *conjugate circles.*

5. Use Descartes' Circle Formula to show that $a' = 2s - 3a$ and therefore $s' = a' + b + c + d = 3s - 4a$. [4]

In the context of this problem, a *circle configuration* is a quadruple of real numbers (a, b, c, d) representing curvatures of mutually tangent circles A, B, C, D. In other words, a circle configuration is a quadruple (a, b, c, d) of real numbers satisfying Descartes' Circle Formula.

The result in problem 5 allows us to compute the curvatures of the six plots removed on day 2. In this case, $(a, b, c, d) = (-1, 2, 2, 3)$, and $s = 6$. For example, one such plot is tangent to both of the circles C and C' centered at $\left(\pm\frac{1}{2}, 0\right)$, and to one of the circles of radius $\frac{1}{3}$ removed on day 1; it is conjugate to the unit circle (the boundary of the original kingdom U). If this new plot's curvature is a, we can write $(-1, 2, 2, 3) \vdash (a, 2, 2, 3)$, and we say that the first circle configuration *yields* the second.

6a. Use the result of problem 5 to compute the curvatures of all circles removed on day 2, and the corresponding values of s'. [2]

6b. Show that by area, 12% of the kingdom is sold on day 2. [1]

6c. Find the areas of the circles removed on day 3. [2]

6d. Show that the plots sold on day 3 have mean curvature of 23. [1]

7. Prove that the curvature of each circular plot is an integer. [3]

Descartes' Circle Formula can be extended by interpreting the coordinates of points on the plane as complex numbers in the usual way: the point (x, y) represents the complex number $x + yi$. On the complex plane, let z_A, z_B, z_C, z_D be the centers of circles A, B, C, D respectively; as before, a, b, c, d are the curvatures of their respective circles. Then Descartes' Extended Circle Formula states

$$(a \cdot z_A + b \cdot z_B + c \cdot z_C + d \cdot z_D)^2 = 2\left(a^2 z_A^2 + b^2 z_B^2 + c^2 z_C^2 + d^2 z_D^2\right).$$

8a. Suppose that A' is a circle conjugate to A with center $z_{A'}$ and curvature a', and $\hat{s} = a \cdot z_A + b \cdot z_B + c \cdot z_C + d \cdot z_D$. Use Descartes' Extended Circle Formula to show that $a' \cdot z_{A'} = 2\hat{s} - 3a \cdot z_A$ and therefore $a' \cdot z_{A'} + b \cdot z_B + c \cdot z_C + d \cdot z_D = 3\hat{s} - 4a \cdot z_A$. [2]

8b. Prove that the center of each circular plot has coordinates $\left(\frac{u}{c}, \frac{v}{c}\right)$ where u and v are integers, and c is the curvature of the plot. [2]

Given a c-triangle T, let a, b, and c be the curvatures of the three arcs bounding T, with $a \leq b \leq c$, and let d be the curvature of the incircle of T. Define the *circle configuration associated with* T to be $\mathcal{C}(T) = (a, b, c, d)$. Define the c-triangle T to be *proper* if $c \leq d$. For example, circles of curvatures -1, 2, and 3 determine two c-triangles. The incircle of one has curvature 6, so it is proper; the incircle of the other has curvature 2, so it is not proper.

Let P and Q be two c-triangles, with associated configurations $\mathcal{C}(P) = (a, b, c, d)$ and $\mathcal{C}(Q) = (w, x, y, z)$. We say that P *dominates* Q if $a \leq w$, $b \leq x$, $c \leq y$, and $d \leq z$. (The term "dominates" refers to the fact that the radii of the arcs defining Q cannot be larger than the radii of the arcs defining P.)

Removing the incircle from T gives three c-triangles, $T^{(1)}$, $T^{(2)}$, $T^{(3)}$, each bounded by the incircle of T and two of the arcs that bound T. These triangles have associated configurations

$$\begin{aligned} \mathcal{C}(T^{(1)}) &= (b, c, d, a'), \\ \mathcal{C}(T^{(2)}) &= (a, c, d, b'), \\ \mathcal{C}(T^{(3)}) &= (a, b, d, c'), \end{aligned}$$

where a', b', and c' are determined by the formula in problem 5.

9. Let P and Q be two proper c-triangles such that P dominates Q. Let $\mathcal{C}(P) = (a, b, c, d)$ and $\mathcal{C}(Q) = (w, x, y, z)$.

a. Show that $P^{(3)}$ dominates $P^{(2)}$ and that $P^{(2)}$ dominates $P^{(1)}$. [2]

b. Prove that $P^{(1)}$ dominates $Q^{(1)}$. [2]

c. Prove that $P^{(3)}$ dominates $Q^{(3)}$. [2]

10a. Prove that the largest plot sold by the king on day n has curvature $n^2 + 2$. [3]

10b. If $\rho = \phi + \sqrt{\phi}$, as in problem 4, prove that the curvature of the smallest plot sold by the king on day n does not exceed $2\rho^n$. [3]

8 Power Solutions

1a. By symmetry, P_1, P_2, the two plots sold on day 1, are centered on the y-axis, say at $(0, \pm y)$ with $y > 0$. Let these plots have radius r. Because P_1 is tangent to U, $y + r = 1$. Because P_1 is tangent to C, the distance from $(0, 1-r)$ to $\left(\frac{1}{2}, 0\right)$ is $r + \frac{1}{2}$. Therefore

$$\begin{aligned} \left(\frac{1}{2}\right)^2 + (1-r)^2 &= \left(r + \frac{1}{2}\right)^2 \\ 1 - 2r &= r \\ r &= \frac{1}{3} \\ y = 1 - r &= \frac{2}{3}. \end{aligned}$$

Thus the plots are centered at $\left(0, \pm\frac{2}{3}\right)$ and have radius $\frac{1}{3}$.

1b. The four "removed" circles have radii $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}$ so the combined area of the six remaining curvilinear territories is:

$$\pi\left(1^2 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right) = \frac{5\pi}{18}.$$

2a. At the beginning of day 2, there are six c-triangles, so six incircles are sold, dividing each of the six territories into three smaller curvilinear triangles. So a total of 18 curvilinear triangles exist at the start of day 3, each of which is itself divided into three pieces that day (by the sale of a total of 18 regions bounded by the territories' incircles). Therefore there are 54 regions at the end of day 3.

2b. Each day, every curvilinear territory is divided into three smaller curvilinear territories. Let R_n be the number of regions at the end of day n. Then $R_0 = 2$, and $R_{n+1} = 3 \cdot R_n$. Thus R_n is a geometric sequence, so $R_n = 2 \cdot 3^n$. For $n > 0$, the number of plots sold on day n equals the number of territories existing at the end of day $n-1$, i.e., $X_n = R_{n-1}$, so $X_0 = X_1 = 2$, and for $n > 1, X_n = 2 \cdot 3^{n-1}$.

2c. The total number of plots sold up to and including day n is

$$\begin{aligned} 2 + \sum_{k=1}^{n} X_k &= 2 + 2\sum_{k=1}^{n} 3^{k-1} \\ &= 2 + 2 \cdot (1 + 3 + 3^2 + \ldots + 3^{n-1}) \\ &= 3^n + 1. \end{aligned}$$

Alternatively, proceed by induction: on day 0, there are $2 = 3^0 + 1$ plots sold, and for $n \geq 0$,

$$\begin{aligned} (3^n + 1) + X_{n+1} &= (3^n + 1) + 2 \cdot 3^n \\ &= 3 \cdot 3^n + 1 \\ &= 3^{n+1} + 1. \end{aligned}$$

3a. Use Descartes' Circle Formula with $a = b = 1$ and $c = \frac{3}{2}$ to solve for d:

$$\begin{aligned} 2 \cdot \left(1^2 + 1^2 + \left(\frac{3}{2}\right)^2 + d^2\right) &= \left(1 + 1 + \frac{3}{2} + d\right)^2 \\ \frac{17}{2} + 2d^2 &= \frac{49}{4} + 7d + d^2 \\ d^2 - 7d - \frac{15}{4} &= 0, \end{aligned}$$

from which $d = \frac{15}{2}$ or $d = -\frac{1}{2}$. These values correspond to radii of $\frac{2}{15}$, a small circle nestled between the other three, or 2, a large circle enclosing the other three.

Alternatively, start by scaling the kingdom with the first four circles removed to match the situation given. Thus the three given circles are internally tangent to a circle of radius $r = 2$ and curvature $d = -\frac{1}{2}$. Descartes' Circle Formula gives a quadratic equation for d, and the sum of the roots is $2 \cdot (1 + 1 + \frac{3}{2}) = 7$, so the second root is $7 + \frac{1}{2} = \frac{15}{2}$, corresponding to a circle of radius $r = \frac{2}{15}$.

3b. Apply Descartes' Circle Formula to yield

$$(a + b + c + x)^2 = 2 \cdot (a^2 + b^2 + c^2 + x^2),$$

a quadratic equation in x. Expanding and rewriting in standard form yields the equation

$$x^2 - px + q = 0$$

where $p = 2(a + b + c)$ and $q = 2(a^2 + b^2 + c^2) - (a + b + c)^2$.

The discriminant of this quadratic is

$$\begin{aligned} p^2 - 4q &= 8(a + b + c)^2 - 8(a^2 + b^2 + c^2) \\ &= 16(ab + ac + bc). \end{aligned}$$

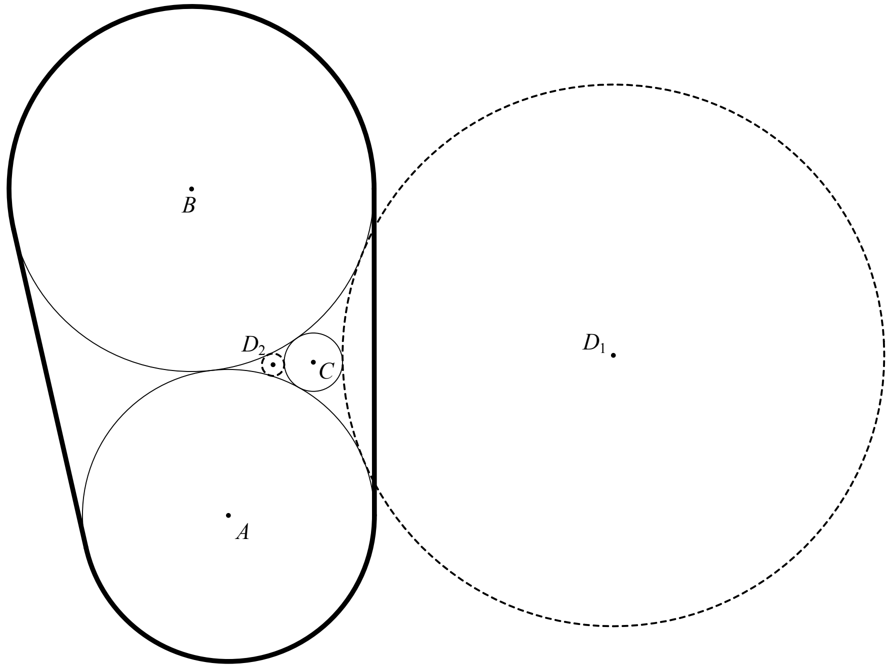
This last expression is positive because it is given that $a, b, c > 0$. Therefore the quadratic has two distinct, real roots, say d_1 and d_2. These usually correspond to two distinct radii, $r_1 = \frac{1}{|d_1|}$ and $r_2 = \frac{1}{|d_2|}$.

There are two possible exceptions. If $(a, b, c, 0)$ satisfies Descartes' Circle Formula, then one of the radii is undefined. The other case to consider is if $r_2 = r_1$, which would occur if $d_2 = -d_1$. This case can be ruled out because $d_1 + d_2 = p = 2(a + b + c)$, which must be positive if $a, b, c > 0$. (Notice too that this inequality rules out the possibility that both circles have negative curvature, so that there cannot be two distinct circles to which the given circles are internally tangent.)

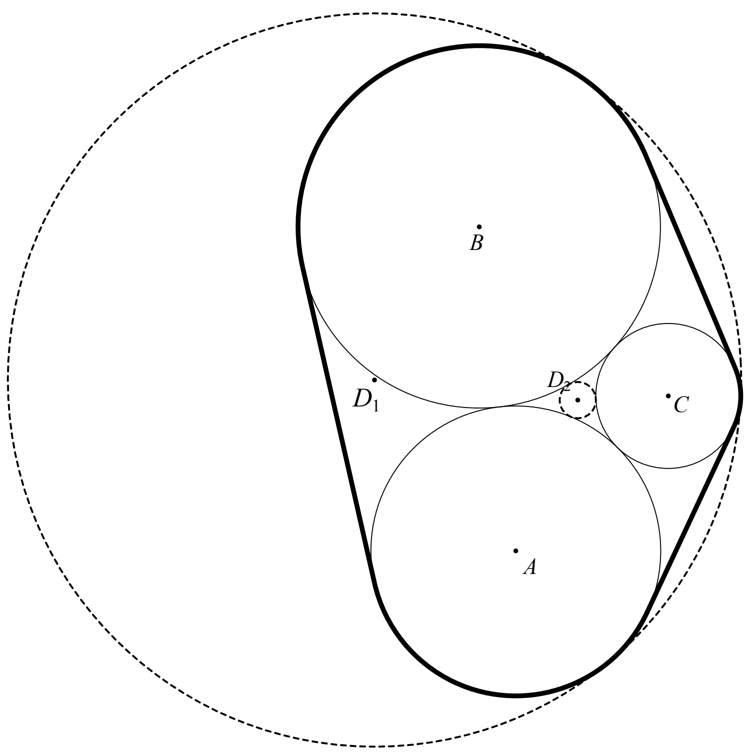
When both roots d_1 and d_2 are positive, the three given circles are externally tangent to both fourth circles. When one is positive and one is negative, the three given circles are internally tangent to one circle and externally tangent to the other.

While the foregoing answers the question posed, it is interesting to examine the result from a geometric perspective: why are there normally two possible fourth circles? Consider the case when one of a, b, and c is negative (i.e., two circles are internally tangent to a third). Let A and B be circles internally tangent to C. Then A and B partition the remaining area of C into two c-triangles, each of which has an incircle, providing the two solutions.

If, as in the given problem, $a, b, c > 0$, then all three circles A, B, and C, are mutually externally tangent. In this case, the given circles bound a c-triangle, which has an incircle, corresponding to one of the two roots. The complementary arcs of the given circles bound an infinite region, and this region normally contains a second circle tangent to the given circles. To demonstrate this fact geometrically, consider shrink-wrapping the circles: the shrink-wrap is the border of the smallest convex region containing all three circles. (This region is called the *convex hull* of the circles). There are two cases to address. If only two circles are touched by the shrink-wrap, then one circle is wedged between two larger ones and completely enclosed by their common tangents. In such a case, a circle can be drawn so that it is tangent to all three circles as shown in the diagram below (shrink-wrap in bold; locations of fourth circle marked at D_1 and D_2).



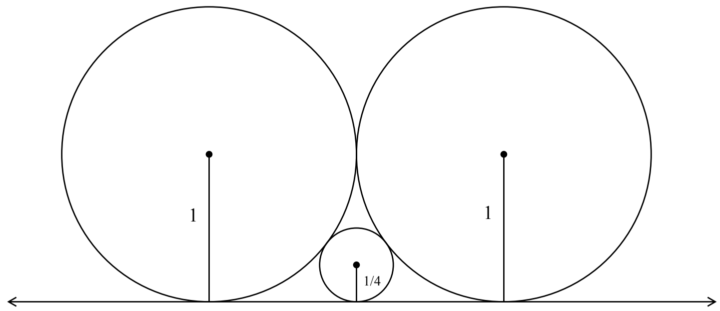
On the other hand, if the shrink-wrap touches all three circles, then it can be expanded to make a circle tangent to and containing A, B, and C, as shown below.



The degenerate case where $(a, b, c, 0)$ satisfies Descartes' Circle Formula is treated in 3c below.

One final question is left for the reader to investigate. Algebraically, it is possible that there is a double root if $p^2 - 4q = 0$. To what geometric situation does this correspond, and under what (geometric) conditions can it arise?

3c. In this case, the fourth "circle" is actually a line tangent to all three circles, as shown in the diagram below.



4a. Note that

$$\begin{aligned}\frac{1}{\rho} &= \frac{\phi^2-\phi}{\phi+\sqrt{\phi}} \\ &= \phi-\sqrt{\phi}.\end{aligned}$$

Therefore

$$\begin{aligned}\left(\rho-\frac{1}{\rho}\right)^2 &= \left(2\sqrt{\phi}\right)^2 = 4\phi \\ &= 2\left(\rho+\frac{1}{\rho}\right).\end{aligned}$$

Multiplying both sides of the equation by ρ^2 gives $\left(\rho^2-1\right)^2 = 2\left(\rho^3+\rho\right)$. Expand and isolate ρ^4 to obtain $\rho^4 = 2\rho^3+2\rho^2+2\rho-1$.

Alternate Proof: Because $\phi^2=\phi+1$, any power of ρ can be expressed as an integer plus integer multiples of $\sqrt{\phi}$, ϕ, and $\phi\sqrt{\phi}$. In particular,

$$\begin{aligned}\rho^2 &= \phi^2+2\phi\sqrt{\phi}+\phi \\ &= 2\phi\sqrt{\phi}+2\phi+1, \\ \rho^3 &= 4\phi\sqrt{\phi}+5\phi+3\sqrt{\phi}+4, \text{ and} \\ \rho^4 &= 12\phi\sqrt{\phi}+16\phi+8\sqrt{\phi}+9.\end{aligned}$$

Therefore

$$\begin{aligned}2\rho^3+2\rho^2+2\rho-1 &= 2\left(4\phi\sqrt{\phi}+5\phi+3\sqrt{\phi}+4\right)+2\left(2\phi\sqrt{\phi}+2\phi+1\right)+2\rho-1 \\ &= 12\phi\sqrt{\phi}+16\phi+8\sqrt{\phi}+9 \\ &= \rho^4.\end{aligned}$$

4b. If the radii are in geometric progression, then so are their reciprocals (i.e., curvatures). Without loss of generality, let $(a,b,c,d) = \left(a, ar, ar^2, ar^3\right)$ for $r>1$. By Descartes' Circle Formula,

$$\left(a+ar+ar^2+ar^3\right)^2 = 2\left(a^2+a^2r^2+a^2r^4+a^2r^6\right).$$

Cancel a^2 from both sides of the equation to obtain

$$\left(1+r+r^2+r^3\right)^2 = 2\left(1+r^2+r^4+r^6\right).$$

Because $1+r+r^2+r^3=(1+r)(1+r^2)$ and $1+r^2+r^4+r^6=(1+r^2)(1+r^4)$, the equation can be rewritten as follows:

$$\begin{aligned}(1+r)^2\left(1+r^2\right)^2 &= 2\left(1+r^2\right)\left(1+r^4\right)\\ (1+r)^2\left(1+r^2\right) &= 2\left(1+r^4\right)\\ r^4-2r^3-2r^2-2r+1 &= 0.\end{aligned}$$

Using the identity from 4a, $r=\rho$ is one solution; because the polynomial is palindromic, another real solution is $r=\rho^{-1}=\phi-\sqrt{\phi}$, but this value is less than 1. The product of the corresponding linear factors is $r^2-2\phi r+\phi^2-\phi=r^2-2\phi r+1$. Division verifies that the other quadratic factor of the polynomial is $r^2+(2\phi-2)r+1$, which has no real roots because $(\sqrt{5}-1)^2<1$.

5. The equation $(x+b+c+d)^2=2\left(x^2+b^2+c^2+d^2\right)$ is quadratic with two solutions. Call them a and a'. These are the curvatures of the two circles which are tangent to circles with curvatures b,c, and d. Rewrite the equation in standard form to obtain $x^2-2\left(b+c+d\right)x+...=0$. Using the sum of the roots formula, $a+a'=2\left(b+c+d\right)=2\left(s-a\right)$. So $a'=2s-3a$, and therefore

$$\begin{aligned}s' &= a'+b+c+d\\ &= 2s-3a+s-a\\ &= 3s-4a.\end{aligned}$$

6. Day 1 starts with circles of curvature $-1,2,2$ bounding C and C'. The geometers mark off P,P' with curvature 3 yielding configurations $(-1,2,2,3),\ s=6$. Then the king sells two plots of curvature 3.

a. To find plots sold on day 2 start with the configuration $(-1,2,2,3)$ and compute the three distinct curvatures of circles conjugate to one of the circles in this configuration. Because $s=6$, the curvatures are $2\cdot6-3(-1)=15$, $2\cdot6-3\cdot2=6$, and $2\cdot6-3\cdot3=3$. However, if P is the circle of curvature 3 included in the orientation, then P' is the new conjugate circle of curvature 3, which was also marked off on day 2. Thus the only options are 15 and 6. In the first case, $s'=3s-4a=3\cdot6-4(-1)=22$; in the second, $s'=3s-4a=3\cdot6-4\cdot2=10$.

b. On day 2, six plots are sold: two with curvature 15 from the configuration $(2,2,3,15)$, and four with curvature 6 from the configuration $(-1,2,3,6)$. The total area sold on day 2 is therefore

$$2\cdot\frac{\pi}{15^2}+4\cdot\frac{\pi}{6^2}=\frac{3}{25}\pi,$$

which is exactly 12% of the unit circle.

c. Day 3 begins with two circles of curvature 15 from the configuration $(2,2,3,15)$, and four circles of curvature 6 from the configuration $(-1,2,3,6)$. Consider the following two cases:

Case 1: $(a,b,c,d)=(2,2,3,15),\ s=22$

- $a=2:a'=2s-3a=\mathbf{38}$
- $b=2:b'=2s-3b=\mathbf{38}$
- $c=3:c'=2s-3c=\mathbf{35}$
- $d=15:d'=2s-3d=-1$, which is the configuration from day 1.

Case 2: $(a,b,c,d)=(-1,2,3,6), s=10$

- $a=-1:a'=2s-3a=\mathbf{23}$
- $b=2\ :b'=2s-3b=\mathbf{14}$
- $c=3\ :c'=2s-3c=\mathbf{11}$
- $d=6\ :d'=2s-3d=2$, which is the configuration from day 1.

So the areas of the plots removed on day 3 are:

$$\frac{\pi}{38^2},\ \frac{\pi}{35^2},\ \frac{\pi}{23^2},\ \frac{\pi}{14^2},\ \text{and}\ \frac{\pi}{11^2}.$$

There are two circles with area $\frac{\pi}{35^2}$, and four circles with each of the other areas, for a total of 18 plots.

d. Because 18 plots were sold on day 3, the mean curvature is $\frac{2(38+38+35)+4(23+14+11)}{18} = 23$.

7. Proceed by induction. The base case, that all curvatures prior to day 2 are integers, was shown in problem 1a. Using the formula $a' = 2s - 3a$, if a, b, c, d, and s are integers on day n, then a', b', c', and d' are integer curvatures on day $n + 1$, proving inductively that all curvatures are integers.

8a. Notice that substituting az_A, bz_B, cz_C, dz_D for a, b, c, d respectively in the derivation of the formula in problem 4 leaves the algebra unchanged, so in general, $a'z_{A'} = 2\hat{s} - 3az_A$, and similarly for $b'z_{B'}, c'z_{C'}, d'z_{D'}$.

8b. It suffices to show that for each circle C with curvature c and center z_C (in the complex plane), cz_C is of the form $u + iv$ where u and v are integers. If this is the case, then each center is of the form $\left(\frac{u}{c}, \frac{v}{c}\right)$.

Proceed by induction. To check the base case, check the original kingdom and the first four plots: U, C, C', P_1, and P_2. Circle U is centered at $(0, 0)$, yielding $-1 \cdot z_U = 0 + 0i$. Circles C and C' are symmetric about the y-axis, so it suffices to check just one of them. Circle C has radius $\frac{1}{2}$ and therefore curvature 2. It is centered at $\left(\frac{1}{2}, \frac{0}{2}\right)$, yielding $2z_C = 2\left(\frac{1}{2} + 0i\right) = 1$. Circles P_1 and P_2 are symmetric about the x-axis, so it suffices to check just one of them. Circle P_1 has radius $\frac{1}{3}$ and therefore curvature 3. It is centered at $\left(\frac{0}{3}, \frac{2}{3}\right)$, yielding $3z_{P_3} = 0 + 2i$.

For the inductive step, suppose that $a\, z_A, b\, z_B, c\, z_C, d\, z_D$ have integer real and imaginary parts. Then by closure of addition and multiplication in the integers, $a'z_{A'} = 2\hat{s} - 3a\, z_A$ also has integer real and imaginary parts, and similarly for $b'z_{B'}, c'z_{C'}, d'z_{D'}$.

So for all plots A sold, $a\, z_A$ has integer real and imaginary parts, so each is centered at $\left(\frac{u}{c}, \frac{v}{c}\right)$ where u and v are integers, and c is the curvature.

9a. Let $s = a + b + c + d$. From problem 5, it follows that

$$\begin{aligned}
\mathcal{C}(P^{(1)}) &= (b, c, d, a'),\\
\mathcal{C}(P^{(2)}) &= (a, c, d, b'),\\
\mathcal{C}(P^{(3)}) &= (a, b, d, c'),
\end{aligned}$$

where $a' = 2s - 3a$, $b' = 2s - 3b$, and $c' = 2s - 3c$. Because $a \le b \le c$, it follows that $c' \le b' \le a'$. Therefore $P^{(3)}$ dominates $P^{(2)}$ and $P^{(2)}$ dominates $P^{(1)}$.

9b. Because $\mathcal{C}(P^{(1)}) = (b, c, d, a')$, and $\mathcal{C}(Q^{(1)}) = (x, y, z, w')$, it is enough to show that $a' \le w'$. As in the solution to problem 5, a and a' are the two roots of the quadratic given by Descartes' Circle Formula:

$$(X + b + c + d)^2 = 2(X^2 + b^2 + c^2 + d^2).$$

Solve by completing the square:

$$\begin{aligned}
X^2 - 2(b + c + d)X + 2(b^2 + c^2 + d^2) &= (b + c + d)^2;\\
\big(X - (b + c + d)\big)^2 &= 2(b + c + d)^2 - 2(b^2 + c^2 + d^2)\\
&= 4(bc + bd + cd).
\end{aligned}$$

Thus a, $a' = b + c + d \pm 2\sqrt{bc + bd + cd}$.

Because $a \le b \le c \le d$, and only a can be less than zero, a must get the minus sign, and a' gets the plus sign:

$$a' = b + c + d + 2\sqrt{bc + bd + cd}.$$

Similarly,

$$w' = x + y + z + 2\sqrt{xy + xz + yz}.$$

Because P dominates Q, each term in the expression for a' is less than or equal to the corresponding term in the expression for w', thus $a' \le w'$.

9c. Because $\mathcal{C}(P^{(3)}) = (a, b, d, c')$, and $\mathcal{C}(Q^{(3)}) = (w, x, z, y')$, it suffices to show that $c' \le y'$. If $a \ge 0$, then the argument is exactly the same as in problem 9a, but if $a < 0$, then there is more to be done.

Arguing as in 9b, $c, c' = a+b+d \pm 2\sqrt{ab + ad + bd}$. If $a < 0$, then the other three circles are *internally* tangent to the circle of curvature a, so this circle has the largest radius. In particular, $\frac{1}{|a|} > \frac{1}{b}$. Thus $b > |a| = -a$, which shows that $a + b > 0$. Therefore c must get the minus sign, and c' gets the plus sign. The same argument applies to y and y'.

When $a < 0$, it is also worth considering whether the square roots are defined (and real). In fact, they are. Consider the diameters of the circles with curvatures b and d along the line through the centers of these circles. These two diameters form a single segment inside the circle with curvature a, so the sum of the diameters is at most the diameter of that circle: $\frac{2}{b} + \frac{2}{d} \le \frac{2}{|a|}$. It follows that $-ad - ab = |a|\, d + |a|\, b \le bd$, or $ab + ad + bd \ge 0$. This is the argument of the square root in the expressions for c and c'. An analogous argument shows that the radicands are real in the expressions for b and b'.

The foregoing shows that

$$c' = a + b + d + 2\sqrt{ab + ad + bd}$$

and

$$y' = w + x + z + 2\sqrt{wx + wz + xz}.$$

It remains to prove that $c' \le y'$. Note that only a and w may be negative; b, c, d, x, y, and z are all positive. There are three cases.

(i) If $0 \le a \le w$, then $ab \le wx$, $ad \le wz$, and $bd \le xz$, so $c' \le y'$.

(ii) If $a < 0 \le w$, then $ab + ad + bd \le bd$, and $bd \le xz \le wx + wz + xz$, so $c' \le y'$. (As noted above, both radicands are positive.)

(iii) If $a \le w < 0$, then it has already been established that $a + b$ is positive. Analogously, $a + d$, $w + x$, and $w + z$ are positive. Furthermore, $a^2 \ge w^2$. Thus $(a+b)(a+d) - a^2 \le (w+x)(w+z) - w^2$, which establishes that $ab + ad + bd \le wx + wz + xz$, so $c' \le y'$.

10a. First, show by induction that every c-triangle on every day in the kingdom is proper. For the base case, both c-triangles at the end of day 0 have configuration $(-1, 2, 2, 3)$, so they are proper. For the inductive step, let T be a proper c-triangle. If $\mathcal{C}(T) = (a, b, c, d)$, then the three c-triangles obtained from T on the next day have configurations $\mathcal{C}(T^{(1)}) = (b, c, d, a')$, $\mathcal{C}(T^{(2)}) = (a, c, d, b')$, and $\mathcal{C}(T^{(3)}) = (a, b, d, c')$. According to problem 5, $c' = 2a + 2b - c + 2d = 2(a + b) + (d - c) + d$. Arguing as in the proof of 9c, $a + b \ge 0$. By inductive hypothesis, T is proper, $d - c \ge 0$; therefore $c' \ge d$. Because $a' \ge b' \ge c'$, all three c-triangles are proper.

If $n = 0$, then both circles sold on day n have curvature 2, and this fits the formula: $n^2 + 2 = 0^2 + 2 = 2$.

Let P_0 be one of the c-triangles left at the end of day 0. For $m > 0$, let $P_m = {P_{m-1}}^{(3)}$. Use induction to prove the following two claims:

(i) $\mathcal{C}(P_m) = (-1, 2, m^2 + 2, (m + 1)^2 + 2)$.

(ii) P_m dominates all c-triangles left at the end of day m.

For the moment, grant these two claims. Then (ii) implies that the incircle of P_{n-1} is at least as large as any plot sold on day n, and (i) shows that this incircle has curvature $n^2 + 2$.

For the base case, both c-triangles at the end of day 0 have associated circle configuration $\mathcal{C}(P_0) = (-1, 2, 2, 3) = (-1, 2, 0^2 + 2, 1^2 + 2)$, so either dominates the other.

For $m > 0$, assume inductively that $\mathcal{C}(P_{m-1}) = (-1, 2, (m-1)^2+2, m^2+2) = (a, b, c, d)$. Because $P_m = {P_{m-1}}^{(3)}$, $\mathcal{C}(P_m) = (a, b, d, c') = (-1, 2, m^2 + 2, c')$. Use algebra and the result of problem 5 to obtain $c' = 2(a + b + c + d) - 3c = (m + 1)^2 + 2$. This completes the inductive step for (i).

Now let Q be any c-triangle left at the end of day $m-1$, with $\mathcal{C}(Q) = (x, y, z, w)$. Any c-triangle left at the end of day m is of the form $Q^{(1)}$, $Q^{(2)}$, or $Q^{(3)}$ for some such Q. By the inductive hypothesis, P_{m-1} dominates Q. It has already been established that these c-triangles are both proper, so the results of problem 9 apply. By 9c, $P_m = {P_{m-1}}^{(3)}$ dominates $Q^{(3)}$, and by 9a, $Q^{(3)}$ dominates $Q^{(1)}$ and $Q^{(2)}$. This completes the inductive step for (ii).

10b. Let R_n be a c-triangle with configuration

$$\mathcal{C}(R_n) = (2\rho^{n-2}, 2\rho^{n-1}, 2\rho^n, 2\rho^{n+1}).$$

According to problem 4, these four numbers (a geometric progression with common ratio ρ) satisfy Descartes' Circle Formula, so there is such a c-triangle. The following inductive argument proves that for $n \geq 2$, each c-triangular plot remaining in the kingdom at the end of day n dominates R_n.

For the base case, problem 6 shows that the following are sufficient: $2 \leq 2\rho^0$, $3 \leq 2\rho^1$, $15 \leq 2\rho^2$, and $38 \leq 2\rho^3$. In fact, it is enough to calculate $\phi = \frac{1+\sqrt{5}}{2} > 1.6$, $\sqrt{\phi} > 1.2$, $\rho = \phi + \sqrt{\phi} > 2.8$ to conclude that $2\rho > 5.6$, $2\rho^2 > 15.68$, and $2\rho^3 > 43.9$. These same calculations show that the main result is true for $n \leq 2$.

For the inductive step, let T be a c-triangle in the kingdom remaining at the end of day n. The inductive hypothesis is that T dominates R_n. By problems 9a and 9b, all the c-triangles obtained from T on day $n+1$ dominate ${R_n}^{(1)}$. All that remains is to show that ${R_n}^{(1)}$ has the right configuration, so that $R_{n+1} = {R_n}^{(1)}$.

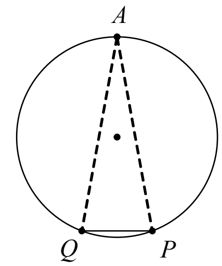
One approach is to use the formulas from problems 4a and 5 to show that $2(2\rho^{n-2} + 2\rho^{n-1} + 2\rho^n + 2\rho^{n+1}) - 3 \cdot 2\rho^{n-2} = 2\rho^{n+2}$. A method that avoids calculation is to note that $\mathcal{C}({R_n}^{(1)}) = (2\rho^{n-1}, 2\rho^n, 2\rho^{n+1}, x)$ for some x. Because Descartes' Circle Formula is quadratic in x, there are at most two possibilities. According to 4b, two solutions are given by $x = 2\rho^{n-2}$ and $x = 2\rho^{n+2}$, because both of these give geometric progressions with common ratio ρ. The first corresponds to R_n, so the second must correspond to ${R_n}^{(1)}$. This completes the induction.

If $n \leq 2$, then (as already noted) the solution to problem 6 shows that the curvature of the smallest plot sold on day n does not exceed $2\rho^n$. If $n > 2$, then this smallest plot is the incircle of some c-triangle T that remains at the end of day $n-1$, with $n-1 \geq 2$. Because T dominates R_{n-1}, the curvature of its incircle does not exceed that of R_{n-1}, which is $2\rho^n$.

9 Relay Problems

Relay 1-1 If A, R, M, and L are positive integers such that $A^2 + R^2 = 20$ and $M^2 + L^2 = 10$, compute the product $A \cdot R \cdot M \cdot L$.

Relay 1-2 Let $T = TNYWR$. A regular n-gon is inscribed in a circle; P and Q are consecutive vertices of the polygon, and A is another vertex of the polygon as shown. If $m\angle APQ = m\angle AQP = T \cdot m\angle QAP$, compute the value of n.



Relay 1-3 Let $T = TNYWR$. Compute the last digit, in base 10, of the sum

$$T^2 + (2T)^2 + (3T)^2 + \ldots + \left(T^2\right)^2.$$

Relay 2-1 A fair coin is flipped n times. Compute the smallest positive integer n for which the probability that the coin has the same result every time is less than 10%.

Relay 2-2 Let $T = TNYWR$. Compute the smallest positive integer n such that there are at least T positive integers in the domain of $f(x) = \sqrt{-x^2 - 2x + n}$.

Relay 2-3 Let $T = TNYWR$. Compute the smallest positive real number x such that $\dfrac{\lfloor x \rfloor}{x - \lfloor x \rfloor} = T$.

10 Relay Answers

Answer 1-1 24

Answer 1-2 49

Answer 1-3 5

Answer 2-1 5

Answer 2-2 35

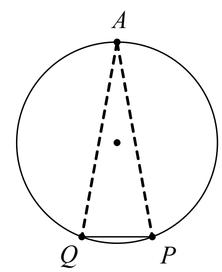
Answer 2-3 $\frac{36}{35}$

11 Relay Solutions

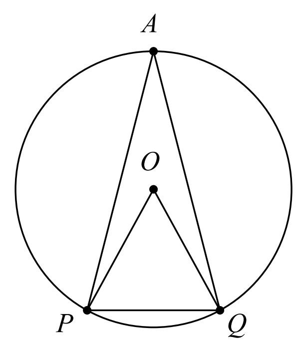
Relay 1-1 If A, R, M, and L are positive integers such that $A^2+R^2=20$ and $M^2+L^2=10$, compute the product $A\cdot R\cdot M\cdot L$.

Solution 1-1 The only positive integers whose squares sum to 20 are 2 and 4. The only positive integers whose squares sum to 10 are 1 and 3. Thus $A\cdot R=8$ and $M\cdot L=3$, so $A\cdot R\cdot M\cdot L=\mathbf{24}$.

Relay 1-2 Let $T=TNYWR$. A regular n-gon is inscribed in a circle; P and Q are consecutive vertices of the polygon, and A is another vertex of the polygon as shown. If $\mathrm{m}\angle APQ=\mathrm{m}\angle AQP=T\cdot\mathrm{m}\angle QAP$, compute the value of n.



Solution 1-2 Let $\mathrm{m}\angle A=x$. Then $\mathrm{m}\angle P=\mathrm{m}\angle Q=Tx$, and $(2T+1)x=180°$, so $x=\frac{180°}{2T+1}$. Let O be the center of the circle, as shown below.



Then $\mathrm{m}\angle POQ=2\mathrm{m}\angle PAQ=2\left(\frac{180°}{2T+1}\right)=\frac{360°}{2T+1}$. Because $\mathrm{m}\angle POQ=\frac{360°}{n}$, the denominators must be equal: $n=2T+1$. Substitute $T=24$ to find $n=\mathbf{49}$.

Relay 1-3 Let $T=TNYWR$. Compute the last digit, in base 10, of the sum

$$T^2+(2T)^2+(3T)^2+\ldots+\left(T^2\right)^2.$$

Solution 1-3 Let S be the required sum. Factoring T^2 from the sum yields

$$\begin{aligned} S &= T^2\left(1+4+9+\ldots+T^2\right) \\ &= T^2\left(\frac{T(T+1)(2T+1)}{6}\right) \\ &= \frac{T^3(T+1)(2T+1)}{6}. \end{aligned}$$

Further analysis makes the final computation simpler. If $T\equiv 0,2,$ or $3 \bmod 4$, then S is even. Otherwise, S is odd. And if $T\equiv 0,2,$ or $4 \bmod 5$, then $S\equiv 0 \bmod 5$; otherwise, $S\equiv 1 \bmod 5$. These observations yield the following table:

$T \bmod 4$	$T \bmod 5$	$S \bmod 10$
$0, 2, 3$	$0, 2, 4$	0
$0, 2, 3$	$1, 3$	6
1	$0, 2, 4$	5
1	$1, 3$	1

Because $T = 49$, the value corresponds to the third case above; the last digit is **5**.

Relay 2-1 A fair coin is flipped n times. Compute the smallest positive integer n for which the probability that the coin has the same result every time is less than 10%.

Solution 2-1 After the first throw, the probability that the succeeding $n-1$ throws have the same result is $\frac{1}{2^{n-1}}$. Thus $\frac{1}{2^{n-1}} < \frac{1}{10} \Rightarrow 2^{n-1} > 10 \Rightarrow n - 1 \geq 4$, so $n = \mathbf{5}$ is the smallest possible value.

Relay 2-2 Let $T = TNYWR$. Compute the smallest positive integer n such that there are at least T positive integers in the domain of $f(x) = \sqrt{-x^2 - 2x + n}$.

Solution 2-2 Completing the square under the radical yields $\sqrt{n + 1 - (x + 1)^2}$. The larger zero of the radicand is $-1 + \sqrt{n+1}$, and the smaller zero is negative because $-1 - \sqrt{n+1} < 0$, so the T positive integers in the domain of f must be $1, 2, 3, \ldots, T$. Therefore $-1 + \sqrt{n+1} \geq T$. Hence $\sqrt{n+1} \geq T + 1$, and $n + 1 \geq (T + 1)^2$. Therefore $n \geq T^2 + 2T$, and substituting $T = 5$ yields $n \geq 35$. So $n = \mathbf{35}$ is the smallest such value.

Relay 2-3 Let $T = TNYWR$. Compute the smallest positive real number x such that $\dfrac{\lfloor x \rfloor}{x - \lfloor x \rfloor} = T$.

Solution 2-3 If $\frac{\lfloor x \rfloor}{x - \lfloor x \rfloor} = T$, the equation can be rewritten as follows:

$$\begin{aligned} \frac{x - \lfloor x \rfloor}{\lfloor x \rfloor} &= \frac{1}{T} \\ \frac{x}{\lfloor x \rfloor} - 1 &= \frac{1}{T} \\ \frac{x}{\lfloor x \rfloor} &= \frac{T+1}{T}. \end{aligned}$$

Now $0 < x < 1$ is impossible because it makes the numerator of the original expression 0. To make x as small as possible, place it in the interval $1 < x < 2$, so that $\lfloor x \rfloor = 1$. Then $x = \frac{T+1}{T}$. When $T = 35$, $x = \mathbf{\frac{36}{35}}$.

12 Super Relay

1. Let N be a perfect square between 100 and 400, inclusive. What is the only digit that <u>cannot</u> appear in N?

2. Let $T = TNYWR$. Let A and B be distinct digits in base T, and let N be the largest number of the form $\underline{A}\,\underline{B}\,\underline{A}_{\,T}$. Compute the value of N in base 10.

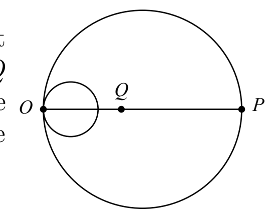
3. Let $T = TNYWR$. Given a nonzero integer n, let $f(n)$ denote the sum of all numbers of the form i^d, where $i = \sqrt{-1}$, and d is a divisor (positive or negative) of n. Compute $f(2T+1)$.

4. Let $T = TNYWR$. Compute the real value of x for which there exists a solution to the system of equations

$$\begin{aligned} x + y &= 0 \\ x^3 - y^3 &= 54 + T. \end{aligned}$$

5. Let $T = TNYWR$. In $\triangle ABC$, $AC = T^2$, $\mathrm{m}\angle ABC = 45°$, and $\sin \angle ACB = \frac{8}{9}$. Compute AB.

6. Let $T = TNYWR$. In the diagram at right, the smaller circle is internally tangent to the larger circle at point O, and $\overline{OP}$ is a diameter of the larger circle. Point Q lies on $\overline{OP}$ such that $PQ = T$, and $\overline{PQ}$ does <u>not</u> intersect the smaller circle. If the larger circle's radius is three times the smaller circle's radius, find the least possible <u>integral</u> radius of the larger circle.



7. Let $T = TNYWR$. The sequence $a_1, a_2, a_3, \ldots$ is an arithmetic progression, d is the common difference, $a_T = 10$, and $a_K = 2010$, where $K > T$. If d is an integer, compute the value of K such that $|K - d|$ is minimal.

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15. Let $f(x) = 2^x + x^2$. Compute the smallest integer $n > 10$ such that $f(n)$ and $f(10)$ have the same units digit.

14. Let $T = TNYWR + 2000$. Given that $\mathsf{A}, \mathsf{D}, \mathsf{E}, \mathsf{H}, \mathsf{S}$, and W are distinct digits, and that $\underline{\mathsf{W}}\,\underline{\mathsf{A}}\,\underline{\mathsf{D}}\,\underline{\mathsf{E}} + \underline{\mathsf{A}}\,\underline{\mathsf{S}}\,\underline{\mathsf{H}} = T$, what is the largest possible value of $\mathsf{D} + \mathsf{E}$?

13. Let $T = TNYWR$. Suppose that a, b, c, and d are real numbers so that $\log_a c = \log_b d = T$. Compute

$$\frac{\log_{\sqrt{ab}}(cd)^3}{\log_a c + \log_b d}.$$

12. Let $T = TNYWR$, and let $K = T + 2$. Compute the largest K-digit number which has distinct digits and is a multiple of 63.

11. Let $T = TNYWR$, and let K be the sum of the digits of T. Let A_n be the number of ways to tile a $1 \times n$ rectangle using 1×3 and 1×1 tiles that do not overlap. Tiles of both types need not be used; for example, $A_3 = 2$ because a 1×3 rectangle can be tiled with three 1×1 tiles or one 1×3 tile. Compute the smallest value of n such that $A_n \geq T$.

10. Let $T = TNYWR$. A cube has volume $T - 2$. The cube's surface area equals one-eighth the surface area of a $2 \times 2 \times n$ rectangular prism. Compute n.

9. Let $T = TNYWR$. In triangle ABC, the altitude from A to $\overline{BC}$ has length $\sqrt{T}$, $AB = AC$, and $BC = T - K$, where K is the real root of the equation $x^3 - 8x^2 - 8x - 9 = 0$. Compute AB.

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8. Let A be the number you will receive from position 7, and let B be the number you will receive from position 9. There are exactly two ordered pairs of real numbers $(x_1, y_1), (x_2, y_2)$ that satisfy both $|x + y| = 6(\sqrt{A} - 5)$ and $x^2 + y^2 = B^2$. Compute $|x_1| + |y_1| + |x_2| + |y_2|$.

13 Super Relay Answers

Answer 1. 7

Answer 2. 335

Answer 3. 0

Answer 4. 3

Answer 5. $8\sqrt{2}$

Answer 6. 9

Answer 7. 49

Answer 8. 24

Answer 9. $6\sqrt{2}$

Answer 10. 23

Answer 11. 10

Answer 12. 98721

Answer 13. 3

Answer 14. 9

Answer 15. 30

Answer to the Super-Relay: 24

14 Super Relay Solutions

Problem 1. Let N be a perfect square between 100 and 400, inclusive. What is the only digit that <u>cannot</u> appear in N?

Solution 1. When the perfect squares between 100 and 400 inclusive are listed out, every digit except **7** is used. Note that the perfect squares 100, 256, 289, 324 use each of the other digits.

Problem 2. Let $T = TNYWR$. Let A and B be distinct digits in base T, and let N be the largest number of the form $\underline{A}\,\underline{B}\,\underline{A}\,_T$. Compute the value of N in base 10.

Solution 2. To maximize $\underline{A}\,\underline{B}\,\underline{A}\,_T$ with $A \neq B$, let $A = T-1$ and $B = T-2$. Then $\underline{A}\,\underline{B}\,\underline{A}_T = (T-1)\cdot T^2 + (T-2)\cdot T^1 + (T-1)\cdot T^0 = T^3 - T - 1$. With $T = 7$, the answer is **335**.

Problem 3. Let $T = TNYWR$. Given a nonzero integer n, let $f(n)$ denote the sum of all numbers of the form i^d, where $i = \sqrt{-1}$, and d is a divisor (positive or negative) of n. Compute $f(2T+1)$.

Solution 3. Let $n = 2^m r$, where r is odd. If $m = 0$, then n is odd, and for each d that divides n, $i^d + i^{-d} = i^d + \frac{i^d}{(i^2)^d} = 0$, hence $f(n) = 0$ when n is odd. If $m = 1$, then for each d that divides n, $i^d + i^{-d}$ equals 0 if d is odd, and -2 if d is even. Thus when n is a multiple of 2 but not 4, $f(n) = -2P$, where P is the number of positive odd divisors of n. Similarly, if $m = 2$, then $f(n) = 0$, and in general, $f(n) = 2(m-2)P$ for $m \geq 1$. Because T is an integer, $2T+1$ is odd, hence the answer is **0**. (Note: If $r = p_1^{a_1} p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$, where the p_i are distinct odd primes, it is well known that $P = (a_1+1)(a_2+1)\ldots(a_k+1)$.)

Problem 4. Let $T = TNYWR$. Compute the real value of x for which there exists a solution to the system of equations

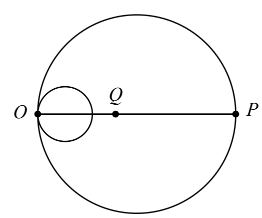
$$\begin{aligned} x + y &= 0 \\ x^3 - y^3 &= 54 + T. \end{aligned}$$

Solution 4. Plug $y = -x$ into the second equation to obtain $x = \sqrt[3]{\frac{54+T}{2}}$. With $T = 0, x = \sqrt[3]{27} = \mathbf{3}$.

Problem 5. Let $T = TNYWR$. In $\triangle ABC$, $AC = T^2$, $m\angle ABC = 45°$, and $\sin\angle ACB = \frac{8}{9}$. Compute AB.

Solution 5. From the Law of Sines, $\frac{AB}{\sin\angle ACB} = \frac{AC}{\sin\angle ABC}$. Thus $AB = \frac{8}{9}\cdot\frac{T^2}{1/\sqrt{2}} = \frac{8\sqrt{2}}{9}\cdot T^2$. With $T = 3$, $AB = \mathbf{8\sqrt{2}}$.

Problem 6. Let $T = TNYWR$. In the diagram at right, the smaller circle is internally tangent to the larger circle at point O, and $\overline{OP}$ is a diameter of the larger circle. Point Q lies on $\overline{OP}$ such that $PQ = T$, and $\overline{PQ}$ does <u>not</u> intersect the smaller circle. If the larger circle's radius is three times the smaller circle's radius, find the least possible <u>integral</u> radius of the larger circle.



Solution 6. Let r be the radius of the smaller circle. Then the conditions defining Q imply that $PQ = T < 4r$. With $T = 8\sqrt{2}$, note that $r > 2\sqrt{2} \to 3r > 6\sqrt{2} = \sqrt{72}$. The least integer greater than $\sqrt{72}$ is **9**.

Problem 7. Let $T = TNYWR$. The sequence $a_1, a_2, a_3, \ldots$ is an arithmetic progression, d is the common difference, $a_T = 10$, and $a_K = 2010$, where $K > T$. If d is an integer, compute the value of K such that $|K - d|$ is minimal.

Solution 7. Note that $a_T = a_1 + (T-1)d$ and $a_K = a_1 + (K-1)d$, hence $a_K - a_T = (K-T)d = 2010 - 10 = 2000$. Thus $K = \frac{2000}{d} + T$, and to minimize $|T + \frac{2000}{d} - d|$, choose a positive integer d such that $\frac{2000}{d}$ is also an integer and $\frac{2000}{d} - d$ is as close as possible to $-T$. Note that $T > 0$, so $\frac{2000}{d} - d$ should be negative, i.e., $d^2 > 2000$ or $d > 44$. The value of T determines how far apart $\frac{2000}{d}$ and d need to be. For example, if T is close to zero, then choose d such that $\frac{2000}{d}$ and d are close to each other. With $T = 9$, take $d = 50$ so that $\frac{2000}{d} = 40$ and $|K - d| = |49 - 50| = 1$. Thus $K = \mathbf{49}$.

Problem 15. Let $f(x) = 2^x + x^2$. Compute the smallest integer $n > 10$ such that $f(n)$ and $f(10)$ have the same units digit.

Solution 15. The units digit of $f(10)$ is the same as the units digit of 2^{10}. Because the units digits of powers of 2 cycle in groups of four, the units digit of 2^{10} is 4, so the units digit of $f(10)$ is 4. Note that n must be even, otherwise, the units digit of $f(n)$ is odd. If n is a multiple of 4, then 2^n has 6 as its units digit, which means that n^2 would need to have a units digit of 8, which is impossible. Thus n is even, but is not a multiple of 4. This implies that the units digit of 2^n is 4, and so n^2 must have a units digit of 0. The smallest possible value of n is therefore **30**.

Problem 14. Let $T = TNYWR + 2000$. Given that $\mathsf{A}, \mathsf{D}, \mathsf{E}, \mathsf{H}, \mathsf{S}$, and W are distinct digits, and that $\underline{\mathsf{W}}\,\underline{\mathsf{A}}\,\underline{\mathsf{D}}\,\underline{\mathsf{E}} + \underline{\mathsf{A}}\,\underline{\mathsf{S}}\,\underline{\mathsf{H}} = T$, what is the largest possible value of $\mathsf{D} + \mathsf{E}$?

Solution 14. First note that if $T \geq 10000$, then $\mathsf{W} = 9$ and $\mathsf{A} \geq 5$. If $T < 10000$ and x is the leading digit of T, then either $\mathsf{W} = x$ and $\mathsf{A} \leq 4$ or $\mathsf{W} = x - 1$ and $\mathsf{A} \geq 5$. With $T = 2030$, either $\underline{\mathsf{W}}\,\underline{\mathsf{A}} = 20$ or $\underline{\mathsf{W}}\,\underline{\mathsf{A}} = 15$. In either case, $\underline{\mathsf{D}}\,\underline{\mathsf{E}} + \underline{\mathsf{S}}\,\underline{\mathsf{H}} = 30$. Considering values of $\mathsf{D} + \mathsf{E}$, there are three possibilities to consider:

$\mathsf{D} + \mathsf{E} = 11$: $\underline{\mathsf{D}}\,\underline{\mathsf{E}} = 29$, $\underline{\mathsf{S}}\,\underline{\mathsf{H}} = 01$, which duplicates digits;
$\mathsf{D} + \mathsf{E} = 10$: $\underline{\mathsf{D}}\,\underline{\mathsf{E}} = 28$, $\underline{\mathsf{S}}\,\underline{\mathsf{H}} = 02$ or $\underline{\mathsf{D}}\,\underline{\mathsf{E}} = 19$, $\underline{\mathsf{S}}\,\underline{\mathsf{H}} = 11$, both of which duplicate digits;
$\mathsf{D} + \mathsf{E} = 9$: $\underline{\mathsf{D}}\,\underline{\mathsf{E}} = 27$, $\underline{\mathsf{S}}\,\underline{\mathsf{H}} = 03$, in which no digits are duplicated if $\underline{\mathsf{W}}\,\underline{\mathsf{A}} = 15$.

Therefore the answer is **9**.

Problem 13. Let $T = TNYWR$. Suppose that a, b, c, and d are real numbers so that $\log_a c = \log_b d = T$. Compute

$$\frac{\log_{\sqrt{ab}}(cd)^3}{\log_a c + \log_b d}.$$

Solution 13. Note that $a^T = c$ and $b^T = d$, thus $(ab)^T = cd$. Further note that $(ab)^{3T} = (\sqrt{ab})^{6T} = (cd)^3$, thus $\log_{\sqrt{ab}}(cd)^3 = 6T$. Thus the given expression simplifies to $\frac{6T}{2T} = \mathbf{3}$ (as long as $T \neq 0$).

Problem 12. Let $T = TNYWR$, and let $K = T + 2$. Compute the largest K-digit number which has distinct digits and is a multiple of 63.

Solution 12. Let N_K be the largest K-digit number which has distinct digits and is a multiple of 63. It can readily be verified that $N_1 = 0, N_2 = 63$, and $N_3 = 945$. For $K > 3$, compute N_K using the following strategy: start with the number $M_0 = \underline{9}\,\underline{8}\,\underline{7}\ldots\underline{(10-K)}$; let M_1 be the largest multiple of 63 not exceeding M_0. That is, to compute M_1, divide M_0 by 63 and discard the remainder: $M_0 = 1587 \cdot 63 + 44$, so $M_1 = M_0 - 44 = 1587 \cdot 63$. If M_1 has distinct digits, then $N_K = M_1$. Otherwise, let $M_2 = M_1 - 63, M_3 = M_2 - 63$, and so on; then N_K is the first term of the sequence $M_1, M_2, M_3, \ldots$ that has distinct digits. Applying this strategy gives $N_4 = 9765, N_5 = 98721, N_6 = 987651$, and $N_7 = 9876510$. With $T = 3$, $K = 5$, and the answer is **98721**.

Problem 11. Let $T = TNYWR$, and let K be the sum of the digits of T. Let A_n be the number of ways to tile a $1 \times n$ rectangle using 1×3 and 1×1 tiles that do not overlap. Tiles of both types need not be used; for example, $A_3 = 2$ because a 1×3 rectangle can be tiled with three 1×1 tiles or one 1×3 tile. Compute the smallest value of n such that $A_n \geq T$.

Solution 11. Consider the rightmost tile of the rectangle. If it's a 1×1 tile, then there are A_{n-1} ways to tile the remaining $1 \times (n-1)$ rectangle, and if it's a 1×3 tile, then there are A_{n-3} ways to tile the remaining $1 \times (n-3)$ rectangle. Hence $A_n = A_{n-1} + A_{n-3}$ for $n > 3$, and $A_1 = A_2 = 1, A_3 = 2$. Continuing the sequence gives the following values:

n	1	2	3	4	5	6	7	8	9	10
A_n	1	1	2	3	4	6	9	13	19	28

With $T = 98721, K = 27$, hence the answer is **10**.

Problem 10. Let $T = TNYWR$. A cube has volume $T - 2$. The cube's surface area equals one-eighth the surface area of a $2 \times 2 \times n$ rectangular prism. Compute n.

Solution 10. The cube's side length is $\sqrt[3]{T}$, so its surface area is $6\sqrt[3]{T^2}$. The rectangular prism has surface area $2(2 \cdot 2 + 2 \cdot n + 2 \cdot n) = 8 + 8n$, thus $6\sqrt[3]{T^2} = 1 + n$. With $T = 8$, $n = 6\sqrt[3]{64} - 1 = \mathbf{23}$.

Problem 9. Let $T = TNYWR$. In triangle ABC, the altitude from A to $\overline{BC}$ has length $\sqrt{T}$, $AB = AC$, and $BC = T - K$, where K is the real root of the equation $x^3 - 8x^2 - 8x - 9 = 0$. Compute AB.

Solution 9. Rewrite the equation as $x^3 - 1 = 8(x^2 + x + 1)$, so that $(x - 1)(x^2 + x + 1) = 8(x^2 + x + 1)$. Because $x^2 + x + 1$ has no real zeros, it can be canceled from both sides of the equation to obtain $x - 1 = 8$ or $x = 9$. Hence $BC = T - 9$, and $AB^2 = (\sqrt{T})^2 + \left(\frac{T-9}{2}\right)^2 = T + \left(\frac{T-9}{2}\right)^2$. Substitute $T = 23$ to obtain $AB = \sqrt{72} = \mathbf{6\sqrt{2}}$.

Problem 8. Let A be the number you will receive from position 7, and let B be the number you will receive from position 9. There are exactly two ordered pairs of real numbers $(x_1, y_1), (x_2, y_2)$ that satisfy both $|x + y| = 6(\sqrt{A} - 5)$ and $x^2 + y^2 = B^2$. Compute $|x_1| + |y_1| + |x_2| + |y_2|$.

Solution 8. Note that the graph of $x^2 + y^2 = B^2$ is a circle of radius $|B|$ centered at $(0, 0)$ (as long as $B^2 > 0$). Also note that the graph of $|x + y| = 6(\sqrt{A} - 5)$ is either the line $y = -x$ if $A = 25$, or the graph consists of two parallel lines with slope -1 if $A > 25$. In the former case, the line $y = -x$ intersects the circle at the points $\left(\pm\frac{|B|}{\sqrt{2}}, \mp\frac{|B|}{\sqrt{2}}\right)$. In the latter case, the graph is symmetric about the origin, and in order to have exactly two intersection points, each line must be tangent to the circle, and the tangency points are $\left(\frac{|B|}{\sqrt{2}}, \frac{|B|}{\sqrt{2}}\right)$ and $\left(-\frac{|B|}{\sqrt{2}}, -\frac{|B|}{\sqrt{2}}\right)$. In either case, $|x_1| + |y_1| + |x_2| + |y_2| = 2\sqrt{2} \cdot |B|$, and in the case where the graph is two lines, this is also equal to $12(\sqrt{A} - 5)$. Thus if $A \neq 25$, then only one of A or B is needed to determine the answer. With $A = 49$ and $B = 6\sqrt{2}$, the answer is $2\sqrt{2} \cdot 6\sqrt{2} = 12(\sqrt{49} - 5) = \mathbf{24}$.

15 Tiebreaker Problems

Problem 1. Let set $S = \{1, 2, 3, 4, 5, 6\}$, and let set T be the set of all subsets of S (including the empty set and S itself). Let $t_1,\ t_2,\ t_3$ be elements of T, not necessarily distinct. The ordered triple (t_1, t_2, t_3) is called *satisfactory* if either

(a) both $t_1 \subseteq t_3$ and $t_2 \subseteq t_3$, or

(b) $t_3 \subseteq t_1$ and $t_3 \subseteq t_2$.

Compute the number of satisfactory ordered triples (t_1, t_2, t_3).

Problem 2. Let $ABCD$ be a parallelogram with $\angle ABC$ obtuse. Let $\overline{BE}$ be the altitude to side $\overline{AD}$ of $\triangle ABD$. Let X be the point of intersection of $\overline{AC}$ and $\overline{BE}$, and let F be the point of intersection of $\overline{AB}$ and $\overleftrightarrow{DX}$. If $BC = 30$, $CD = 13$, and $BE = 12$, compute the ratio $\frac{AC}{AF}$.

Problem 3. Compute the sum of all positive two-digit factors of $2^{32} - 1$.

16 Tiebreaker Answers

Answer 1. 31186

Answer 2. $\frac{222}{13}$

Answer 3. 168

17 Tiebreaker Solutions

Problem 1. Let set $S = \{1, 2, 3, 4, 5, 6\}$, and let set T be the set of all subsets of S (including the empty set and S itself). Let $t_1,\ t_2,\ t_3$ be elements of T, not necessarily distinct. The ordered triple (t_1, t_2, t_3) is called *satisfactory* if either

(a) both $t_1 \subseteq t_3$ and $t_2 \subseteq t_3$, or

(b) $t_3 \subseteq t_1$ and $t_3 \subseteq t_2$.

Compute the number of satisfactory ordered triples (t_1, t_2, t_3).

Solution 1. Let $T_1 = \{(t_1, t_2, t_3) \mid t_1 \subseteq t_3 \text{ and } t_2 \subseteq t_3\}$ and let $T_2 = \{(t_1, t_2, t_3) \mid t_3 \subseteq t_1 \text{ and } t_3 \subseteq t_2\}$. Notice that if $(t_1, t_2, t_3) \in T_1$, then $(S \setminus t_1, S \setminus t_2, S \setminus t_3) \in T_2$, so that $|T_1| = |T_2|$. To count T_1, note that if $t_1 \subseteq t_3$ and $t_2 \subseteq t_3$, then $t_1 \cup t_2 \subseteq t_3$. Now each set t_3 has $2^{|t_3|}$ subsets; t_1 and t_2 could be any of these, for a total of $\left(2^{|t_3|}\right)^2 = 4^{|t_3|}$ possibilities given a particular subset t_3. For $n = 0, 1, \ldots, 6$, if $|t_3| = n$, there are $\binom{6}{n}$ choices for the elements of t_3. So the total number of elements in T_1 is

$$\begin{aligned} |T_1| &= \sum_{k=0}^{6} \binom{6}{k} 4^k \\ &= (4+1)^6 = 15625, \end{aligned}$$

by the Binomial Theorem. However, $T_1 \cap T_2 \neq \emptyset$, because if $t_1 = t_2 = t_3$, the triple (t_1, t_2, t_3) satisfies both conditions and is in both sets. Therefore there are 64 triples that are counted in both sets. So $|T_1 \cup T_2| = 2 \cdot 15625 - 64 = \mathbf{31186}$.

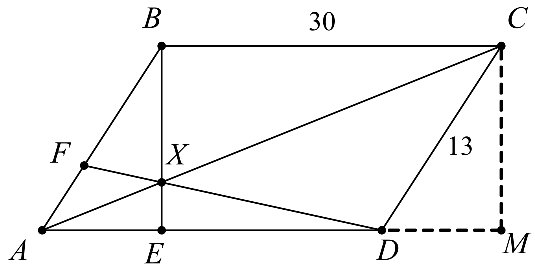
Alternate Solution: Let T_1 and T_2 be defined as above. Then count $|T_1|$ based on the number n of elements in $t_1 \cup t_2$. There are $\binom{6}{n}$ ways to choose those n elements. For each element a in $t_1 \cup t_2$, there are three possibilities: $a \in t_1$ but not t_2, or $a \in t_2$ but not t_1, or $a \in t_1 \cap t_2$. Then for each element b in $S \setminus (t_1 \cup t_2)$, there are two possibilities: either $b \in t_3$, or $b \notin t_3$. Combine these observations in the table below:

$\lvert t_1 \cup t_2 \rvert$	Choices for $t_1 \cup t_2$	Ways of dividing between t_1 and t_2	$\lvert S \setminus (t_1 \cup t_2) \rvert$	Choices for t_3	Total
0	1	1	6	2^6	64
1	6	3	5	2^5	576
2	15	3^2	4	2^4	2160
3	20	3^3	3	2^3	4320
4	15	3^4	2	2^2	4860
5	6	3^5	1	2^1	2916
6	1	3^6	0	2^0	729

The total is 15625, so $|T_1| = |T_2| = 15625$. As noted in the first solution, there are 64 triples that are counted in both T_1 and T_2, so $|T_1 \cup T_2| = 2 \cdot 15625 - 64 = \mathbf{31186}$.

Problem 2. Let $ABCD$ be a parallelogram with $\angle ABC$ obtuse. Let $\overline{BE}$ be the altitude to side $\overline{AD}$ of $\triangle ABD$. Let X be the point of intersection of $\overline{AC}$ and $\overline{BE}$, and let F be the point of intersection of $\overline{AB}$ and $\overleftrightarrow{DX}$. If $BC = 30$, $CD = 13$, and $BE = 12$, compute the ratio $\frac{AC}{AF}$.

Solution 2. Extend $\overline{AD}$ to a point M such that $\overline{CM}||\overline{BE}$ as shown below.



Because $CD = AB = 13$ and $BE = 12 = CM$, $AE = DM = 5$. Then $AC = \sqrt{35^2 + 12^2} = \sqrt{1369} = 37$. Because $\overline{EX}||\overline{CM}$, $XE/CM = AE/AM = \frac{1}{7}$. Thus $EX = \frac{12}{7}$ and $XB = \frac{72}{7}$, from which $EX/XB = \frac{1}{6}$. Apply Menelaus's Theorem to $\triangle AEB$ and Menelaus line $\overline{FD}$:

$$\begin{aligned} \frac{AD}{ED} \cdot \frac{EX}{XB} \cdot \frac{BF}{FA} &= 1 \\ \frac{30}{25} \cdot \frac{1}{6} \cdot \frac{13 - FA}{FA} &= 1 \\ \frac{13 - FA}{FA} &= 5. \end{aligned}$$

Thus $FA = \frac{13}{6}$. The desired ratio is:

$$\frac{37}{13/6} = \mathbf{\frac{222}{13}}.$$

Alternate Solution: After calculating AC as above, draw $\overline{BD}$, intersecting $\overline{AC}$ at Y. Because the diagonals of a parallelogram bisect each other, $DY = YB$. Then apply Ceva's Theorem to $\triangle ABD$ and concurrent cevians $\overline{AY}, \overline{BE}, \overline{DF}$:

$$\begin{aligned} \frac{AE}{ED} \cdot \frac{DY}{YB} \cdot \frac{BF}{FA} &= 1 \\ \frac{5}{25} \cdot 1 \cdot \frac{13 - FA}{FA} &= 1. \end{aligned}$$

Thus $FA = \frac{13}{6}$, and the desired ratio is $\mathbf{\frac{222}{13}}$.

Problem 3. Compute the number of two-digit factors of $2^{32} - 1$.

Solution 3. Using the difference of squares, $2^{32} - 1 = \left(2^{16} - 1\right)\left(2^{16} + 1\right)$. The second factor, $2^{16} + 1$, is the Fermat prime 65537, so continue with the first factor:

$$\begin{aligned} 2^{16} - 1 &= \left(2^8 + 1\right)\left(2^8 - 1\right) \\ 2^8 - 1 &= \left(2^4 + 1\right)\left(2^4 - 1\right) \\ 2^4 - 1 &= 15 = 3 \cdot 5. \end{aligned}$$

Because the problem does not specify that the two-digit factors must be prime, the possible two-digit factors are 17, $3 \cdot 17 = 51$, $5 \cdot 17 = 85$ and $3 \cdot 5 = 15$, for a sum of $17 + 51 + 85 + 15 = \mathbf{168}$.