Circles 1

1. As shown in the diagram, points M and N are the midpoints of arcs AB and AC respectively. Prove that $AP = AQ$.

Solution:

We are required to prove that $AP = AQ \Leftrightarrow \angle APQ = \angle AQP$

$\angle APQ = \frac{1}{2}(\widehat{AN} + \widehat{BM})$, $\angle AQP = \frac{1}{2}(\widehat{AM} + \widehat{CN})$

$\because \widehat{AN} = \widehat{CN}$, $\widehat{AM} = \widehat{BM}$

$\therefore \angle APQ = \angle AQP$

2. It is known that AB is a diameter of circle O while AC is a chord. AD, angle bisector of $\angle BAC$ intersects circle O at point D. $DE \perp AB$ at E. Prove that $AC = AE - BE$.

Solution:

Let F be a point on AE such that $EF = BE$.

We are required to prove that $AC = AE - BE \Leftrightarrow AC = AF \Leftrightarrow \Delta ACD \cong \Delta AFD$

$\angle CAD = \angle DAB$

$AD = AD$

$\angle C + \angle DBF = 180°$ ($ACDB$ concyclic) $\angle DFA + \angle DFB = 180°$ $\because \angle DFB = \angle DBF$ $\therefore \angle C = \angle DFA$

$\therefore \Delta ACD \cong \Delta AFD$

3. As shown in the diagram, AB is a diameter of circle O and chord $CD \perp AB$. Let E be a point on CD. AE and BE are extended to intersect the circle at F and G respectively. Prove that $\frac{GC}{FC} = \frac{GD}{FD}$.

Solution:

We are required to prove that $\frac{GC}{FC} = \frac{GD}{FD} \Leftrightarrow \frac{GC}{GD} = \frac{FC}{FD}$

$\frac{GC}{GD} = \frac{CE}{ED}$ ($\widehat{CB} = \widehat{BD}$, $\angle CGB = \angle DGB$, GB is an angle bisector of $\angle DGC$)

Similarly, $\frac{FC}{FD} = \frac{CE}{ED}$.

$\therefore \frac{GC}{GD} = \frac{FC}{FD}$

4. If P, Q, R are the midpoints on the arcs BC, AC, AB on the circumcircle of $\triangle ABC$ respectively, join PQ, PR to intersect AC, AB at points E, D respectively. I is the incenter of of $\triangle ABC$. Prove that D, I, E are collinear and $DE \parallel BC$.

Solution:

$\widehat{BP} = \widehat{PC}$, $\angle BAP = \angle CAP$, AP is an angle bisector of $\angle BAC$ $\therefore$ AIP collinear

Similarly, RIC collinear

Join DI and EI

$\angle PRC = \angle PAC = \angle PAB$ $\therefore$ $RDIA$ concyclic

$\angle AID = 180° - \angle ARD = 180° - \angle B - \frac{1}{2}\angle A$ ($\because \angle ARD = \frac{1}{2}(\widehat{AC} + \widehat{CP})$)

Similarly, $\angle AIE = 180° - \angle C - \frac{1}{2}\angle A$

$\angle AID + \angle AIE = 360° - \angle A - \angle B - \angle C = 180°$

$\angle ADI = \angle ARI$, $\angle ARI = \angle ABC$ $\therefore$ $\angle ADI = \angle ABC$ $\therefore$ $DIE \parallel BC$

5. It is known that P is a point outside circle O. PA, PC intersect the circle at points A, C respectively. Secant PBD intersects the circle at B, D. Prove that $AD \times BC = AB \times CD$.

Solution:

We are required to prove that $\frac{AD}{AB} = \frac{CD}{BC}$

$\frac{AD}{AB} = \frac{PA}{PB}, \frac{CD}{BC} = \frac{PC}{PB}$

$\because PA = PC$ $\frac{PA}{PB} = \frac{PC}{PB} \therefore \frac{AD}{AB} = \frac{CD}{BC}$

6. As shown in the diagram, BC is a diameter of circle O and AD intersects the circle at D. $FE \perp AB$ at E and the area of $\triangle AEF$ is equal to the area of $\triangle ABC$. Prove that $AE = AD$.

Solution:

We are required to prove that $AE = AD \Leftrightarrow AE^2 = AM \times AB$ ($\because AD^2 = AM \times MB$)

Area of $\triangle$AEF = Area of $\triangle$ABC $\Leftrightarrow \frac{1}{2}\text{AE} \times \text{EF} = \frac{1}{2}\text{AB} \times \text{CM} \Leftrightarrow \text{AE} = \text{AB} \times \frac{CM}{EF}$

$\Leftrightarrow \text{AE} = \text{AB} \times \frac{AM}{AE} \therefore AE^2 = AB \times AM$

7. In right-angle triangle ABC, $\angle C = 90°$, $AC = b$, $BC = a$. Construct a circle which passes through D, the midpoint of BC, and which is tangent to AB at E, the midpoint of AB. Find the area of the circle.

Solution:

$\triangle$OFE ~ $\triangle$ACB

EF = $\frac{b}{4}$, EO = r

S = $\frac{b^2(a^2+b^2)\pi}{16a^2}$

8. I is the incenter of $\triangle ABC$ and E is a point on AI extended. AE intersects the circumcircle of $\triangle ABC$ at D and $ID = DE$. Also, $BM \perp AE$ at M. Prove that $\frac{DM}{BE} \times \frac{DI}{BI} = \frac{1}{2}$.

Solution:

We are required to prove that $\frac{BM}{BE} \times \frac{DI}{BI} = \frac{1}{2} \Leftrightarrow BM \times DI = \frac{1}{2} BE \times BI \Leftrightarrow \angle IBE = 90°$ ($\because BM \times DI$ = 2 × area of ΔBDI = area of ΔBEI) $\Leftrightarrow BD = DI \Leftrightarrow \angle DBI = \angle DIB$

$\angle BID = \frac{1}{2}(\angle A + \angle B)$

$\angle DBC = \angle DAC = \frac{1}{2}\angle A$

$\angle DBI = \frac{1}{2}\angle B + \frac{1}{2}\angle A$

$\therefore \angle DBI = \angle DIB$

9. Construct tangents t_1, t_2 of the circumcircle of $\triangle ABC$ which pass through corners B, C of $\triangle ABC$ respectively. Construct $AD \parallel t_2$ which intersects BC at D and $AE \parallel t_1$ which intersects BC at E. Prove that $\frac{BE}{CD} = \frac{AB^2}{AC^2}$.

Solution:

We are required to prove that $\frac{BE}{CD} = \frac{AB^2}{AC^2}$.

$\angle 1 = \angle 2 = \angle 3$

$\Delta BAE \sim \angle BCA$

$AB^2 = BE \times BC$

$AC^2 = CD \times CB$

$\frac{AB^2}{AC^2} = \frac{BE \times BC}{CD \times CB} = \frac{BE}{CD}$

- End of Circles 1 -