Congruence (Part 1)

1. Prove that $3333^{4444} + 4444^{3333}$ is divisible by 7. Make up some more problem of this type.

Solution:

$3333^{4444} + 4444^{3333} \equiv 1^{4444} + (-1)^{3333} \pmod{7} = 1 + (-1) = 0$ (proven)

2. Prove that (a) $8 \mid (55^{1991} + 9)$. (b) $8 \mid (3^{2n} + 7)$. (c) $17 \mid (19^{1000} - 1)$.

Solution:

(a) $55^{1991} + 9 \equiv (-1)^{1991} + 1 \pmod{8} = (-1) + 1 = 0$ (proven)

(b) $3^{2n} + 7 = 9^n + 7 \equiv 1^n - 1 \pmod{8} = 0$ (proven)

(c) $19^{1000} - 1 = (19^4)^{250} - 1 \equiv (-1)^{250} - 1 \pmod{17} = 1 - 1 = 0$ (proven)

3. Let a, b be positive integers with $a = 7q_1 + 2$, $b = 7q_2 + 5$ where q_1, q_2 are integers. Find $a^2 + 4b \bmod 7$ and $a^2 - 4b \bmod 7$.

Solution:

$a^2 + 4b \equiv 2^2 + 4(5) \pmod{7} = 4 + 20 = 24 \equiv \mathbf{3} \pmod{7}$

$a^2 - 4b \equiv 2^2 - 4(5) \pmod{7} = 4 - 20 = -16 \equiv \mathbf{5} \pmod{7}$

4. Given that the last three digits in the decimal representation of 7^{400} are 0, 0, 1 so that $7^{400} = 10^3k + 1$ for some positive integer k, find the last three digits in the decimal representation of 7^{9999}.

Solution:

$7^{9999} = 7^3 \times 7^{9996} = 343\,(7^4)^{2499} = 343\,(2401)^{2499} = 343\,(1 + 2400)^{2499} \equiv 343\,(1 + 2499 \times 2400) \pmod{1000} = 343\,[1 + (2500 - 1)(2400)] \equiv 343\,(1 - 2400) \pmod{1000} \equiv (343)(601) \pmod{1000} = 206\mathbf{143}$

5. (a) Find the last digit of 7^{7^7}.

   (b) Find the last two digits of $7^{7^{7^{\cdot^{\cdot^{7}}}}}$ (k's 7) where $k > 1$, $k \in \mathbb{Z}^+$.

Solution:

(a) $7^7 \equiv (-1)^7 \pmod 4 = -1 \equiv 3 \pmod 4$

Since $7^1 \equiv 7 \pmod{10}$, $7^2 \equiv 9 \pmod{10}$, $7^3 \equiv 3 \pmod{10}$, $7^4 \equiv 1 \pmod{10}$, $7^5 \equiv 7 \pmod{10}$, $7^6 \equiv 9 \pmod{10}$, $7^7 \equiv 3 \pmod{10}$, $7^8 \equiv 1 \pmod{10}$, …, last digit of 7^{7^7} = **3**

(b) $7^{7^{7^{\cdots^7}}}$ $(k-1)$'s 7 $\equiv (-1)^{7^{7^{\cdots^7}}} \pmod 4 = -1 \equiv 3 \pmod 4$

Since $7^1 \equiv 07 \pmod{100}$, $7^2 \equiv 49 \pmod{100}$, $7^3 \equiv 43 \pmod{100}$, $7^4 \equiv 01 \pmod{100}$, $7^5 \equiv 07 \pmod{100}$, $7^6 \equiv 49 \pmod{100}$, $7^7 \equiv 43 \pmod{100}$, $7^8 \equiv 01 \pmod{100}$, …, last two digits of $7^{7^{7^{\cdots^7}}}$ = **43**

6. Find the last three digits of $n = 1 \times 3 \times 5 \times \ldots \times 1997 \times 1999$.

Solution:

$n = 125\, m$

$1000 = 8 \times 125$

$m = (1 \times 3 \times 7)\,(9 \times 11 \times 13 \times 15)\,(17 \times 19 \times 21 \times 23)\,(27 \times 29 \times 31)\,(33 \times 35 \times 37 \times 39) \ldots (1985 \times 1987 \times 1989 \times 1991)\,(1993 \times 1995 \times 1997 \times 1999)$

$(8k + 1)\,(8k + 3)\,(8k + 5)\,(8k + 7) \equiv 1 \times 3 \times 5 \times 7 \equiv 1 \pmod 8$

$1 \times 3 \times 5 \times 7 \equiv 5 \pmod 8$

$27 \times 29 \times 31 \equiv 1 \pmod 8$

Thus $m \equiv 5 \times 1 \equiv 5 \pmod 8$

$m = 8k + 5$

$\therefore 125m = 125\,(8k + 5) \equiv$ **625** $\pmod{1000}$

7. Find the last digit in the septenary (base-7) representation of $47^{37^{27}}$.

Solution:

$37^{27} \equiv 1^{27} \pmod 4 = 1$

$47^{37^{27}} \equiv (-2)^{37^{27}} \pmod 7 \equiv -2 \pmod 7 \equiv$ **5** $\pmod 7$

8. Let n be a positive integer. Prove that $7 \nmid (4^n + 1)$.

Solution:

When $n = 3k$, $4^n + 1 = 4^{3k} + 1 = 64^k + 1 \equiv 1^k + 1 \pmod 7 = 2$

When $n = 3k + 1$, $4^n + 1 = 4^{3k+1} + 1 = 4(64^k) + 1 \equiv 4(1^k) + 1 \pmod 7 = 5$

When $n = 3k + 2$, $4^n + 1 = 4^{3k+2} + 1 = 16(64^k) + 1 \equiv 2(1^k) + 1 \pmod 7 = 3$ (proven)

9. Determine all positive integers n for which $2^n + 1$ is divisible by 3.

Solution:

$2^n + 1 \equiv (-1)^n + 1 \pmod 3$

RHS = 0 when $(-1)^n = -1$, which is when ***n* is odd**.

10. Prove that among 11, 111, 1111, …, 11…11 (n times), … has no perfect squares.

Solution:

Let $2n + 1$ be an odd positive integer.

$(2n + 1)^2 = 4n^2 + 4n + 1 \equiv 1 \pmod 4$

However, 11, 111, 1111, …, 11…11, … $\equiv 3 \pmod 4$ (proven)

11. Prove that $5^{2n+1} + 11^{2n+1} + 17^{2n+1}$ is divisible by 33 for every non-negative integer n.

Solution:

$5^{2n+1} + 11^{2n+1} + 17^{2n+1} = 5(25^n) + 11(121^n) + 17(289^n)$

$5(25^n) + 11(121^n) + 17(289^n) \equiv 2(1^n) + 2(1^n) + 2(1^n) \pmod 3 = 6 \equiv 0 \pmod 3$

$5(25^n) + 11(121^n) + 17(289^n) \equiv 5(3^n) + 6(3^n) \pmod 3 = 11(3^n) \equiv 0 \pmod{11}$ (proven)

12. Prove that if p is prime, then $p^2 \equiv 1 \pmod{24}$ for $p \geq 5$.

Solution:

For $p > 3$, we have $p = 6n \pm 1$

Thus $p^2 = (6n \pm 1)^2 = 36n^2 \pm 12n + 1 = 24n^2 + 12n\,(n \pm 1) + 1$

Since $2 \mid n\,(n + 1)$ and $2 \mid (n - 1)\,n$

$p^2 = 24k + 1$ ($k \in \mathbb{Z}$)

$p^2 \equiv 1 \pmod{24}$ (proven)

13. (a) Prove that if $n > 2$, then $2^n - 1$ is not a power of 3.
(b) Prove that if $n > 3$, then $2^n + 1$ is not a power of 3.

Solution:

(a) Suppose $n > 2$, we want to show that never $2^n - 1 = 3m$ by contradiction.

For odd m, we have $2^n = 3^m + 1 = (3 + 1)\,(3^{m-1} - 3^{m-2} + \ldots + 1)$

The last factor is an odd number of odd summands. Contradiction!

Next suppose $m = 2s$ is even. Then $2^n = 1 + 3^{2s} = 9^s + 1 = 8q + 2$ Contradiction! because it is not a multiple of 4

(b) Suppose $n > 3$. For m odd, we get $2^n = 3^m - 1 = (3 - 1)\,(3^{m-1} + 3^{m-2} + \ldots + 1)$

The last factor is an odd number of odd summands. Contradiction!

Next suppose $m = 2s$ is even. Then $3^s = 2a + 1$, $2n + 1 = (3^s)^2$

$2^n = (2a + 1)^2 - 1 = 4a\,(a + 1)$ Hence a or $a + 1$ is odd. Thus $a = 1$, $2^n = 3^2 - 1$

Hence, no solution for $n > 3$

- End of Congruence (Part 1) -