Menelaus and Ceva Theorems

1.

Solution:

2.

Solution:

3.

Solution:

4. In the quadrilateral ABCD, the ratios of the triangles ABD, BCD and ABC are 3:4:1. Points M and N are on AC and AD respectively, satisfying AM: AC=CN: CD. Also, points B, M, N are collinear. Prove that M and N are the midpoints of AC and CD respectively.

Solution:

Let AM/AC= CN/CD= x

AM=xAC

AE=3/7 AC

EM= AM-AE= (x-3/7)AC

MC= AC-AM=1-x

Using Menelaus theorem, CN/CD × DB/BE × EM/AC=1

x/(1-x) × 7 × (x-2/7)/(1-x)=1

x=1/2

5. It is known that X and X`, Y and Y`, Z and Z` are on the sides BC, CA and AB of triangle ABC. Besides, the six points reside on the same circle. Also, lines AX, BY and CZ intersect at a common point. Prove that AX`, BY`, CZ` either intersect at a common point or are parallel.

Solution:

By Menelaus theorem, BX/XC × CY/YA × AZ/ZB=1

Because, triangle BXZ is similar to BZ`X and triangle CYX` is similar to triangle CXY`, therefore BX/BZ-BZ`/BX`, CY/CX= CX`/CY`, AZ/AY= AY'/AZ`.

Subbing in the values yields BX`/XC × CY`/Y`A × AZ`/Z`B=1

By the converse of ceva's theorem, the proof is complete.

6. In triangle ABC and A`B`C`, lines AA`, BB`, CC` intersect at a common point O. BC and B`C` intersects at P, CA and C`A` intersects at Q, CA and C`A` intersects at R. Prove that P, Q and R are collinear.

Solution:

Using Menelaus theorem, we obtain

BR/RA × AA`/A`O × OB`/B`B=1

AQ/QC × CC`/C`O × OA`/A`A=1

CP`/PB × BB`/B`O × OC`/C`C=1

Multiplying the three equations yield AQ/QC × OP/PB × BR/RA=1

7. Let the intersection of the incircle of the triangle with its three sides, BC, CA and AB be points D, E and F respectively. EF and BC intersects at point P, FD and CA intersects at point Q, DE and AB intersects at point R. Prove that points P, Q and R are collinear.

Solution:

Using Menelaus theorem,

AF/FB × BP/PC × CF/EA=1, because AF=AE, therefore BP/PC=FB/CE

AQ/QC × CD/DB × BF/FA=1, because BF=BD, therefore AQ/QC=FA/CD

$AR/RB \times BD/DC \times CE/EA=1$, because $CE=DC$, therefore $AR/RB=EA/BD$

Multiplying the three equations yield $AR/RB \times BP/PC \times CQ/QA= EA/BD \times FB/CE \times CD/FA=1$

By the converse of Menelaus theorem, the proof is complete.

8.

Solution:

9. a) In triangle ABC, heights AD, BE and CF intersect orthocenter H. Prove that angle EDH= angle FDH
   b) In triangle ABC, AD is the height on line BC. H is a random point on AD. BH intersects AC at E, CH intersects AB at F. Prove that angle EDH= angle FDH
   c) In the quadrilateral ABCD, AD bisects angle BDC. H is a random point on AD. BH intersects AC at E, CH intersects AB at F. Prove that angle EDH= angle FDH

Solution:

a) Points E, B, D, H are concylic. Therefore, angle EDH= angle EBH

Points F, C, D, H are concylic. Therefore, angle FDH= angle FCH

angle EBH+ angle BAC=90^0, angle FCH+ angle BAC=90^0

Therefore, angle EBH= angle FCH, and angle EDH= angle FDH.

b) Extend lines DE and DF to point M and N respectively where line MAN is parallel to line BC.

Triangle DEB is similar to triangle MEA, therefore $AE/EB=MA/BD$

Triangle DCF is similar to triangle NAF, therefore $CF/AF=CD/AN$

Using ceva's theorem, we have $AE/EB \times BD/DC \times CF/FA=1$

Subbing in the above values, we transform the equation to

$MA/BD \times BD/DC \times CD/AN=1$

Simplifying, we obtain MA=MN and subsequently triangle DAM is congruent to triangle DAN. Thus, angle EDH= angle FDH and the proof is completed.

c) Extend lines DE and DF to points M and N respectively such that AM//BD, AN//DC

AE/EB=AM/BD, CF/AF=CD/AN

AE/EB × BD`/D`C × CF/AF=1, BD`/D`C=BD/DC

AM/BD × BD/DC × CD/AN=1, therefore AM= AN

angle MAD+ angle BDA= 180^0, angle NAD+ angle CDA= 180^0

Therefore angle MAD= angle NAD

Thus, triangle MAD is congruent to triangle NAD, and subsequently, angle MDA= angle NDA.

- End of Menelaus and Ceva Theorems – YES!