Number Theory – Congruence

Question 1

Since $3333 \equiv 1 \pmod{7}$ and $4444 \equiv -1 \pmod{7}$, so

$3333^{4444} + 4444^{3333} \equiv 1^{4444} + (-1)^{3333} \equiv 1 - 1 \equiv 0 \pmod{7}$

Question 2 (1)

$55 \equiv -1 \pmod{8}$

Hence, $55^{1991} + 9 \equiv 1^{1991} + 1 \equiv 1 - 1 \equiv 0 \pmod{8}$

Question 2 (2)

$3^{2n} = 9^{n} \equiv 1^{n} \equiv 1 \pmod{8}$

Hence, $3^{2n} + 7 \equiv 1 - 1 \equiv 0 \pmod{8}$

Question 2 (3)

Since $19 \equiv 2 \pmod{17}$, $19^{4} \equiv 16 \equiv -1 \pmod{17}$, so $19^{8} \equiv 1 \pmod{17}$

Hence, $19^{1000} - 1 \equiv (19^{8})^{125} - 1 \equiv 1 - 1 \equiv 0 \pmod{17}$

Question 3

$a^2 + 4b = (7q_1 + 2)^2 + 4(7q_2 + 5)$

$\equiv 2^2 + 20 \equiv 0 \pmod{7}$

$a^2 - 4b = (7q_1 + 2)^2 - 4(7q_2 + 5)$

$\equiv 2^2 - 20 \equiv 5 \pmod{7}$

Question 4

$7^2 = 49$, $7^3 = 343$ and $7^4 = 2401$

$7^{9999} = 7^3 \times 7^{9996}$

$= 343\,(7^{4\times 2499})$

$= 343\,(2401)^{2499}$

$= 343\,(2400 + 1)^{2499}$

$\equiv 343\,(1 + 2400 \times 2499) \pmod{1000}$

$= 343\,(1 + (2500 - 1)\,(2400))$

$\equiv 343\,(1 - 2400) \pmod{1000}$

$\equiv (343)\,(601) \pmod{1000}$

$= 206143 \equiv 143 \pmod{1000}$

Question 5 (1)

$7^4 \quad = 2401$

$\equiv 1 \pmod{10}$

$7^4 \quad \equiv (-1)^7 \pmod{4}$

$\equiv -1 \pmod{4}$

$\equiv 3 \pmod{4}$

Hence, $7^{7^7} = 7^{4k+3}$

$= (7^4)^k \times 7^3$

$\equiv 3 \pmod{10}$

Question 5 (2)

$7^4 \equiv 01 \pmod{100}$

$7^7 \equiv (-1)^7 \pmod 4$

$\equiv -1 \pmod 4$

$\equiv 3 \pmod 4$

Hence, $7^7 = 7^{7^7} = 7^{7^{7^7}} = \ldots = 7^{7^{\cdot^{\cdot^{\cdot^7}}}} = 7^{4k+3}$

Therefore, $7^{7^{\cdot^{\cdot^{\cdot^7}}}} = 7^{4k+3} = (7^k) \times 343 \equiv 43 \pmod{100}$

Question 6

$n = 1 \times 3 \times 5 \ldots\ldots \times 1997 \times 1999 = 125m$ and $1000 = 8 \times 125$

Thus,

$m = (1 \times 3 \times 7)\ (9 \times 11 \times 13 \times 15)\ (17 \times 19 \times 21 \times 23)\ (27 \times 29 \times 31)\ (33 \times 35 \times 37 \times 39) \ldots\ldots$

$(1993 \times 1995 \times 1997 \times 1999)$

$(8k+1)(8k+3)(8k+5)(8k+7) \equiv 1 \times 3 \times 5 \times 7 \equiv 0 \pmod 8$

Also, $1 \times 3 \times 7 \equiv 5 \pmod 8$, $27 \times 29 \times 31 \equiv 1 \pmod 8$

Thus, $m \equiv 5 \pmod 8$

$n = 125m = 125(8k+5) = 1000k + 625 \equiv 625 \pmod{1000}$

Question 7

By, Fermat's Little Theorem, $47^6 \equiv 1 \pmod 7$

$37^{27} \equiv 1^{27} \equiv 1 \pmod 7$

i.e. $37^{27} = 6k + 1$

Hence, $47^{37^{27}} = 47^{6k+1} \equiv 1 \times 47 \equiv 5 \pmod 7$

Question 8

$4^3 = 64 \equiv 1 \pmod 7$

When $n = 3k$, $4^n + 1 \equiv 1 + 1 \equiv 2 \pmod 7$

When $n = 3k + 1$, $4^n + 1 \equiv 4 + 1 \equiv 5 \pmod 7$

When $n = 3k + 3$, $4^n + 1 \equiv 2 + 1 \equiv 3 \pmod 7$

Thus, for all n, $7 \nmid 4^n + 1$

Question 9

If $n = 2k + 1$, $2^n + 1 = 2^{2k+1} + 1$

$$\equiv 2 + 1 \equiv 0 \pmod 3$$

If $n = 2k$, $2^n + 1 = 2^{2k} + 1$

$$\equiv 1 + 1 \equiv 2 \pmod 3$$

Thus, the solutions are all odd numbers.

Question 10

$x^2 \equiv 0, 1 \pmod 4$

$$\underbrace{11\ldots1}_{n-2}11 = \underbrace{11\ldots1}_{n-2}00 + 11$$

$$\equiv 11 \equiv 3 \pmod 4$$

Thus, there is no perfect square.

Question 11

$2n+1$ is odd

$5^{2n+1} \equiv (-1)^{2n+1} \equiv -1 \pmod 3$

Similarly, $11^{2n+1} \equiv -1 \pmod 3$ and $17^{2n+1} \equiv -1 \pmod 3$

Thus, $5^{2n+1} + 11^{2n+1} + 17^{2n+1} \equiv 0 \pmod 3$

Hence, $3 \mid 5^{2n+1} + 11^{2n+1} + 17^{2n+1}$

$11^{2n+1} \equiv 0 \pmod{11}$

$5^{2n+1} \equiv -6^{2n+1} \pmod{11}$ and $17^{2n+1} \equiv 6^{2n+1} \pmod{11}$

Thus, $5^{2n+1} + 11^{2n+1} + 17^{2n+1} \equiv 0 \pmod{11}$

Since 3 and 11 are coprime, so $33 \mid 5^{2n+1} + 11^{2n+1} + 17^{2n+1}$

Question 12

For $p > 3$, we have $p = 6n \pm 1$

Thus, $p^2 = (6n \pm 1)^2 = 36n^2 \pm 12n + 1 = 24n^2 \pm 12n(n \pm 1) + 1$

Since $2 \mid n(n \pm 1)$, so $p^2 = 24r + 1$ $(r \in Z^+) \Rightarrow p^2 \equiv 1 \pmod{24}$

Question 13 (1)

Suppose $n > 2$, we want to show that never $2^n - 1 = 3^m$

For odd m, we have $2^n = 3^m + 1 = (3+1)(3^{m-1} - 3^{m-2} + 3^{m-3} - \ldots\ldots - 3 + 1)$

The last factor is an odd number of odd summands, this is a contradiction.

Let suppose $m = 2s$. Then $2^n = 3^{2s} + 1 = 9^s + 1 = 8q + 2$

Since it is not multiple of 4, this is a contradiction.

Question 13 (2)

Supposed $n > 3$, we want to show that $2^n + 1 \neq 3^m$

For odd n, we have $2^n = 3^m - 1 = (3 - 1)(3^{m-1} + 3^{m-2} + 3^{m-3} - \ldots\ldots + 3 + 1)$

The last factor is an odd number of odd summands, this is a contradiction.

Let suppose $m = 2s$, then $3^s = 2a + 1$

So $2^n = (2a + 1)^2 - 1 = 4a^2 + 4a + 1 - 1 = 4(a)(a + 1)$

Hence a or $a + 1$ is odd

Thus, $a = 1$, $2^n = 3^2 - 1$. Hence, there is no solution

Question 14

$14^{14} \equiv (4^2)^7 \equiv 6 \pmod{10}$ and

$14^{10} \equiv (-4)^5 \equiv 1024 \equiv 7 \pmod{10}$

So, $14^{14^{14}} \equiv 14^{10t+6} \equiv 76 \times (-4)^3 \equiv 76 \times 36 \equiv 36 \pmod{10}$

Question 15

If $3 \mid m$ then $m^2 \equiv 0 \pmod 3$, if $3 \nmid m$ then $m^2 \equiv 1 \pmod 3$

Suppose $3 \nmid a$ with $3 \nmid b$, then

$$a^2 \equiv 1 \pmod 3,\ b^2 \equiv \pmod 3$$

Thus, $c^2 \equiv a^2 + b^2 \equiv 2 \pmod 3$ --- (1)

Also, $c^2 \equiv 0 \pmod 3$ or $c^2 \equiv 1 \pmod 3$ --- (2)

It follows that (1) and (2) contradiction.

Question 16

If $n = 3k$, then $3k \cdot 2^{3k} + 1 \equiv 1 \pmod 3$

If $n = 3k + 1$, then $(3k+1) \cdot 2^{3k+1} + 1 \begin{cases} \equiv 0, & k \text{ is even} \\ \equiv 2, & k \text{ is odd} \end{cases}$

If $n = 3k + 2$, then $(3k+2) \cdot 2^{3k+2} + 1 \begin{cases} \equiv 0, & k \text{ is even} \\ \equiv 2, & k \text{ is odd} \end{cases}$

Thus, the possible values of n is $6k + 1$ and $6k + 2$ where k is all nonnegative integer

Question 17 (a)

Suppose none of the number is divisible by 3 ($z^2 \equiv 0 \pmod 3$ or $z^2 \equiv 1 \pmod 3$)

Then, $1 + 1 \equiv 1$, which is a contradiction.

Question 17 (b)

Suppose none of the x, y, z is divisible by 4.

Suppose x and z are odd and $y = 4q+2$.

Then, we have $1 + 4 \equiv 1 \pmod 4$. This is a contradiction.

Question 17 (c)

Suppose none of the number is divisible by 5.

Then, we have $\pm 1 \pm 1 \equiv \pm 1 \pmod 5$, contradiction.

Question 18

Since $x^2 + 2y^2$ is odd, x must be odd and thus $x^2 \equiv 1 \pmod 8$.

Also, $y^2 \equiv 0, 1, 4 \pmod 8$

Hence, $x^2 + 2y^2 \equiv 1^2 + 2 \times 0^2, 1^2 + 2 \times 1^2, 1^2 + 2 \times 4^2 \pmod 8$

$\equiv 1 \text{ or } 3 \pmod 8$

So, $x^2 + 2y^2$ can be expressed as $8n + 1$ or $8n+3$.

Question 19

$k^4 \equiv 0$ or 1 (mod 5).

Thus, $x^4 + y^4 + 2 \equiv 2, 3$ or 4 (mod 5).

$\therefore\ 5 \nmid x^4 + y^4 + 2$

Question 20

$x^2 + y^2 = 2003^2$, then $x, y < 5$

5 is prime, G.C.D(x, 5) =1 and G.C.D(y, 5) =1

Thus, (x, y, 5) 是方程的本原解

Thus, there exist m, n such that $m^2 + n^2 = 2003$ ($m, n \in Z^+$)

$$x^2 + y^2 \equiv 0, 1 \text{ or } 2 \pmod 4$$

However, $2003 \equiv 3 \pmod 4$

This is a contradiction, so there is no solution for x, y.

Question 21

Let $X = x^3$ and $Y = y^4$,

Taking mod 13, we get

$X + Y \equiv 7 \pmod{13}$ [Since $19^{19} \equiv 7 \pmod{13}$]

Since 13 is prime, by Fermat's Little Theorem,

We have $X^4 \equiv Y^3 \equiv 1 \pmod{13}$

$X^4 - 1 \equiv (X^2 + 1)(X + 1)(X - 1)$

$\equiv (X^2 - 13X + 40)(X - 12)(X - 1)$

$\equiv (X - 12)(X - 1)(X - 8)(X - 5) \equiv 0 \pmod{13} \Rightarrow X \equiv 1, 5, 8, 12 \pmod{13}$

Also,

$Y^3 - 1 \equiv (Y^2 + Y + 1)(Y - 1)$

$\equiv (Y^2 + Y + -12)(Y - 1)$

$\equiv (Y + 4)(Y - 3)(Y - 1)$

$\equiv (X - 6)(X - 4)(X - 11)$ [Since $Y = 7 - X$]

$\equiv \pmod{13}$

which give $X \equiv 4, 6, 11 \pmod{13}$, a contradiction. Thus, there is no solution.

Question 22

Since $2x^2 - 5y^2 = 7$, y must be odd, so $y^2 \equiv 1 \pmod 8$

When x is odd, $x^2 \equiv 1 \pmod 8$

$2x^2 - 5y^2 \equiv 7 \pmod 8$

$2 - 5 \equiv 7 \pmod 8$, a contradiction

When x is even, $2x^2 \equiv 0 \pmod 8$

$2x^2 - 5y^2 \equiv 7 \pmod 8$

$0 - 5 \equiv 7 \pmod 8$, a contradiction

Thus, there is no integer solution for x, y.

Question 23

For any integer a, we have $a^2 \equiv 0 \pmod 4$ or $a^2 \equiv 1 \pmod 8$

It follows that $a^4 \equiv 0$ or $1 \pmod{16}$

If $n < 15$,

Let $x_1^4 + x_2^4 + x_3^4 + \cdots + x_n^4 \equiv m \pmod{16}$, then $m \leq n < 15$.

Since $1599 \equiv 15 \pmod{16}$, contradiction.

Hence, $n \geq 15$.

When $n = 15$, $x_1^4 \equiv x_2^4 \equiv x_3^4 \equiv \cdots \equiv x_n^4 \equiv 1 \pmod{16}$ i.e. $x_1^4,\ x_2^4,\ x_3^4, \ldots, x_n^4$ are odd

$x_1^4 + x_2^4 + x_3^4 + \cdots + x_n^4 = 5^4 + 12 \times 3^4 + 2 \times 1^4 = 1599$

Question 24

$$
\begin{aligned}
p &\equiv 3 \pmod{5} \\
&\equiv 5 \pmod{8} \\
&\equiv 11 \pmod{13}
\end{aligned}
$$

Thus, $p+2 \equiv 0 \pmod{5}$

$\equiv 0 \pmod{13}$

$\equiv 0 \pmod{65}$

Also, $p+2 \equiv 7 \pmod{8}$

$p+67 \equiv 0 \pmod{65}$

$\equiv 0 \pmod{8}$

Hence $p+67 \equiv 0$ (mod L.C.M (5, 8, 13))

$\equiv 0 \pmod{520}$

$p \equiv 453 \pmod{520}$

Therefore, $p_{max} < 1000$ is $520+453=973$.

Question 25

Let k be such a number that $7 \mid 2 \cdot 3^{6n} + k \cdot 2^{3n+1} - 1$ for any n.

i.e. $2 \cdot 3^{6n} + k \cdot 2^{3n+1} - 1 \equiv 0 \pmod 7$

Since $3^6 \equiv 1 \pmod 7$, we have $2 \cdot 3^{6n} \equiv 2 \pmod 7$.

Thus, $k \cdot 2^{3n+1} \equiv -1 \pmod 7$

On the other hand, since $8 \equiv 1 \pmod 7$ and thus $k \cdot 2^{3n+1} \equiv 2k \pmod 7$

We conclude that $2k \equiv -1 \pmod 7$, so we see that 7 is a divisor of $2k + 1$

i.e. $k = \frac{7m-1}{2}$ for some integer m

But the value of m for which k is an integer between 0 and 50, as can be easily seen are

m = 1, 3, 5, 7, 9, 11, 13 and so k = 3, 10, 17, 24, 31, 38, 45.

There are 7 solutions.

Question 26

No.

Suppose they exist, then b+1 and c+1 are even integers, so b and c are odd.

[Use Vieta’s theorem]

$b^2 - 4c \equiv 1 - (-4) \equiv 5 \pmod 8$ is not a square. Thus, this is a contradiction.

Question 27

Question 28

Since N is odd, so $N^2 \equiv 1 \pmod 8$, $N^4 \equiv 1 \pmod 5$

Hence, $N^4 \equiv 1 \pmod{5 \times 8}$, $N^4 = 40k + 1$ ($k \in \mathrm{N}$)

$N^{100} = (40k+1)^{25}$

$= \binom{25}{25}(40k)^{25} + \binom{25}{24}(40k)^{24} + \binom{25}{23}(40k)^{23} + \cdots + \binom{25}{2}(40k)^2 + \binom{25}{1}(40k)^1 + 1$

$\equiv 1 \pmod{1000}$

Thus, $N^{100} = N^{101} \times N$

$\equiv N \pmod{1000}$

Question 29

If $x \equiv 1 \pmod 4$, then $(-1)^n - 1 \equiv 1 + (-1)^n \pmod 4$, which is impossible.

If $x \equiv 2 \pmod 4$, then $1 - (-1)^n \equiv 1 + (-1)^n \pmod 4$, again impossible.

If x is even, then $(x+2)^n - x^n$ is divisible by 2^n

However, $1 + 7^n \equiv 2 \pmod 4$, if n is even and $1 + 7^n \equiv 8 \pmod{16}$ when n is odd.

So 2^n divides $1 + 7^n$ only when n is 1 or 3

Clearly, n cannot be 1. When $n = 3$, we have $6x^2 + 12x + 8 = 1 + 343$

Thus, $x^2 + 2x - 56 = 0$

Since the discriminant is 228, which is not a perfect square, there is no integer solution

Question 30

Suppose, on the contrary that m and n are positive integer such that $m > 5$ and $(m - 1)! = m^n - 1$

Since $m > 5$, then $(m - 1)!$ Is even and thus m is odd.

Therefore, 2, $\frac{m-1}{2}$, m-1 are distinct factors of $(m - 1)!$, and it follows that the product

$$(m - 1)^2 \mid m^n - 1 = (m-1)(m^{n-1} + m^{n-2} + \ldots + m + 1) \Rightarrow m^{n-1} + m^{n-2} + \ldots + m + 1 \equiv 0 \pmod{m - 1}$$

Since $m \equiv 1 \pmod{m - 1}$, $n \equiv 0 \pmod{m - 1}$

Thus, $n \geq m - 1$ (since $n > 0$)

Therefore, $(m - 1)! + 1 < m^{m-1} \leq m^n$ [Since $(m - 1)(m - 2)\ldots.(2)(1) < m^{m-1}$]

This is a contradiction and the proof is complete.

Question 31 (1)

$30 = 2 \times 3 \times 5$

$a^5 - a$

$= a (a^4 - 1)$

$= a (a^2 - 1) (a - 1) (a + 1)$

In a, $(a - 1)$ and $(a + 1)$, one of them is divisible by 2 or 3

By Fermat's Little Theorem, $a^4 - 1 \equiv 0 \pmod 5$

Thus, $30 \mid a^5 - a$

Question 31 (2)

$2730 = 2 \times 3 \times 5 \times 7 \times 13$

By Fermat's Little Theorem,

$n^{13} \equiv n \pmod{13}$, $n^7 \equiv n \pmod 7$, $n^5 \equiv n \pmod 5$, $n^3 \equiv n \pmod 3$, $n^2 \equiv n \pmod 2$

$n^{13} - n = \underline{n(n^{12} - 1)}$

$= n(n^6 - 1)(n^6 + 1) = n\,\underline{(n^7 + n)}$

$= n((n^4)^3 - 1) = \underline{n(n^4 - 1)}\,((n^4)^2 + n^4 - 1)$

$= \underline{n(n^2 - 1)}\,(n^2 + 1)((n^4)^2 + n^4 - 1)$

$= \underline{n(n - 1)}\,(n + 1)(n^2 + 1)((n^4)^2 + n^4 - 1)$

Thus, we have $n^7 - n \mid n^{13} - n$, $n^5 - n \mid n^{13} - n$, $n^3 - n \mid n^{13} - n$ and $n^2 - n \mid n^{13} - n$.

Therefore, $k \mid n^{13} - n$, $k = 2, 3, 5, 7, 13$

But 2, 3, 5, 7, 13 are coprime, so $2730 \mid n^{13} - n$

Question 32

By Fermat's Little Theorem,

$a^{m-1} \equiv 1 \pmod m$ and $b^{m-1} \equiv 1 \pmod m$

$a^{m-1} - b^{m-1} = (a-b)(a^{m-2} + a^{m-3}b + a^{m-4}b^2 + \ldots + a\,b^{m-3} + b^{m-2}) \equiv 1 - 1 \equiv 0 \pmod m$

Since G.C.D. $(a, m) = 1$ and G.C.D. $(b, m) = 1$, then G.C.D. $(a-b, m) = 1$

Thus, $a^{m-2} + a^{m-3}b + a^{m-4}b^2 + \ldots + a\,b^{m-3} + b^{m-2} \equiv 0 \pmod m$

Question 33

$1001 = 7 \times 11 \times 13$

$10^3 \equiv -1 \pmod{13}$, so $10^6 \equiv 1 \pmod{13}$

Assume that $10^k \equiv 4 \pmod{6}$, where k is positive integer

$10^{k+1} \equiv 10^k \times 10 \pmod{6}$

$\equiv 4 \times 4 \pmod{6}$

$\equiv 4 \pmod{6}$

Thus, $10^k \equiv 4 \pmod{6}$ for all k

$10^{10^{10}} \equiv 10^{10m+4}$, where m is a nonnegative integer

$\equiv 10^4 \pmod{13}$

$\equiv -10 \pmod{13}$

$\equiv 3 \pmod{13}$

Hence, $10^{10} + 10^{10^2} + \cdots + 10^{10^{10}} \equiv 3 \times 10 \pmod{13}$

$\equiv 4 \pmod{13}$

Question 34

$1^{241} + 2^{241} + 3^{241} + 4^{241}$

$\equiv 1^{241} + 2^{241} + (-2)^{241} + (-1)^{241}$

$\equiv 0 \pmod 5$

$1^{240} + 2^{240} + 3^{240} + 4^{240}$

$\equiv 1^{240} + 2^{240} + (-2)^{240} + (-1)^{240}$

$\equiv 1^{240} + 2^{240} + 2^{240} + 1^{240}$

$\equiv 2^{241}$

$\equiv 4^{120} \times 2$

$\equiv (-1)^{120} \times 2$

$\equiv 2 \pmod 5$

Question 35

$n^{4q+r} - n^r = n^r (n^{4q} - 1)$

$= n^r [(n^4)^q - 1]$

Factor of $(n^4)^q - 1$ is $n^4 - 1$ (i.e. $(n^4)^q - 1 = (n^4 - 1)(\ldots\ldots)$)

Hence, factor of $n^{4q+r} - n^r$ is $n(n^4 - 1)$

By Fermat's Little Theorem, $5 \mid n(n^4 - 1)$ and $2 \mid n(n^4 - 1)$

Since $n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = n(n - 1)(n + 1)(n^2 + 1)$

G.C.D. (2, 5) =1, so that $10 \mid n(n^4 - 1)$.

Thus, $10 \mid n^{4q+r} - n^r$

Question 36

Since $p > 5$, p is prime. Then, $p \nmid 10$.

By Fermat's Little Theorem,

$10^{p-1} \equiv 1 \pmod{p}$

$10^{l(p-1)} - 1 \equiv 0 \pmod{p}$

$$10^{l(p-1)} - 1 = \underbrace{99\ldots\ldots999}_{l(p-1)}$$

$$10^{l(p-1)} - 1 = 9 \times \underbrace{11\ldots\ldots111}_{l(p-1)}$$

Since $p \nmid 9$ with $p \mid 10^{l(p-1)} - 1$

Question 37

G.C.D. $(a, 17) = 1$

$a^{16} \equiv 1 \pmod{17}$

$a^{16} - 1 \equiv 0 \pmod{17}$

$(a^8 + 1)(a^8 - 1) \equiv 0 \pmod{17}$

Thus, only one of $a^8 + 1$ or $a^8 - 1$ is divisible by 17

Question 38

Question 39

Question 40

If p =2, then taking n is even, we have $p \mid 2^n - n$

Let p >2, then $(2, p) = 1$

By Fermat's Little Theorem, $2^{p-1} \equiv 1 \pmod{p}$

Hence, for any positive integer k, we have $2^{k(p-1)} \equiv 1 \pmod{p}$

Thus, $k(p-1) \equiv 1 \pmod{p}$ [Putting $n = k(p-1)$, then $p \mid 2^n - n$]

$k \equiv -1 \pmod{p}$

Since there are infinitely k, so there are infinitely n

Question 41

By Fermat's Little Theorem,

$p^{q-1} \equiv 1 \pmod{q} \Rightarrow q^{p-1} + p^{q-1} \equiv 1 \pmod{q}$

$q^{p-1} \equiv 1 \pmod{p} \Rightarrow q^{p-1} + p^{q-1} \equiv 1 \pmod{p}$

Thus, $q^{p-1} + p^{q-1} \equiv 1 \pmod{pq}$, and so the remainder is 1.

Question 42

$10^4 \equiv 4 \pmod{7}$

By Fermat's Little Theorem, $10^6 \equiv 1 \pmod{7}$

Thus, for positive integer k, $10^{6k+4} + 3 \equiv 10^4 + 3 \equiv 4 + 3 \equiv 0 \pmod{7}$

Therefore, $10^{6k+4} + 3$, $k = 0, 1, 2, 3 \ldots$ ($k \in \mathrm{N}$) are composite numbers

Question 43

Since $a_n = \frac{1}{3}(10^n - 7)$,

$10^2 \equiv 15 \equiv -2 \pmod{17}$

$10^8 \equiv 16 \equiv -1 \pmod{17}$

Thus, $10^9 \equiv -10 \equiv 7 \pmod{17}$

Also, $10^{16} \equiv 1 \pmod{17}$

$10^{16k+9} \equiv 7 \pmod{17}$ where $k = 0, 1, 2, 3 \ldots$ ($k \in \mathrm{N}$)

Hence, $17 \mid \frac{1}{3}(10^{16k+9} - 7)$

Also, $\frac{1}{3}(10^{16k+9} - 7) \geq \frac{1}{3}(10^9 - 7) > 17$

Therefore, $\frac{1}{3}(10^{16k+9} - 7)$, $k = 1, 2, 3 \ldots$ ($k \in \mathrm{N}$) are composite numbers

Question 44

$2^p - 1 \quad = (2-1)(2^{p-1} + 2^{p-2} + 2^{p-3} + \ldots\ldots + 2 + 1)$

$= 2^{p-1} + 2^{p-2} + 2^{p-3} + \ldots\ldots + 2 + 1$

$= 2(2^{p-2} + 2^{p-3} + 2^{p-4} \ldots\ldots + 2 + 1) + 1$

$= 2(2^{p-1} - 1) + 1$

Since $2^{p-1} \equiv 1 \pmod{p}$, so $2^{p-1} - 1 \equiv 0 \pmod{p}$

Thus, $2^{p-1} = 2px + 1$

Question 45

If $a_1 = 2$, $a_2 = 5$, $a_3 = 11$, $a_4 = 23$, $a_5 = 47$, $a_6 = 95$

a_6 is not prime.

If a_1 is odd prime, then

$a_2 = 2a_1 + 1$,

$a_3 = 2a_2 + 1 = 2(2a_1 + 1) + 1 = 2^2 a_1 + 2^2 - 1$

$\boxed{a_k = 2^{k-1} a_1 + 2^{k-1} - 1, \text{k} > 0}$ [Prove by induction]

Putting $k = a_1 = p$ is an odd prime,

Then, $a_p = 2^{p-1} a_1 + 2^{p-1} - 1$, p >0

By Fermat's Little Theorem, $2^{p-1} - 1 \equiv 0 \pmod{p} \Rightarrow p \mid a_p$

But $p < a_p$, thus a_p is composite number.

Question 46

$2^{2n} + 1$ is odd, so $(2^{2n} + 1)^2 \equiv 1 \pmod 3$

$(2^{2n} + 1)^2 + 2 \equiv 1 + 2 \equiv 0 \pmod 3$

Question 47

$2n + 1$ is an odd perfect square

$2n + 1 = (2p + 1)^2 = 4p^2 + 4p + 1$

Thus, $n = 2p(p + 1) \Rightarrow 4 \mid n$

On the other hand,

$3n + 1 = (2q + 1)^2$

$3(4r) + 1 = 4q^2 + 4q + 1$

$3r = q(q + 1) \Rightarrow 2 \mid r$ and $8 \mid n$

n	$2n + 1$	$3n + 1$
0	1	1
1	3	4
2	0	2
3	2	0
4	4	3

(mod 5)

Since $k^2 \equiv 0, 1, 4 \pmod 5$, thus $n \equiv 0 \pmod 5$

Therefore, $40 \mid n$

Question 48

If n is prime number, then $n \mid m$, $n \mid m^{n-1} +1$

When G.C.D. (m, n) =1, by Fermat's Little Theorem,

We have $n \nmid m^{n-1} +1$

If n is composite number,

Let t ($t \in \mathbb{N}$) such that t is the largest possible number for $2^t \mid n -1$

Also, there exist m, so that $n \mid m^{n-1} +1$

Then, $(m^k)^{2^t} \equiv -1 \pmod{n}$ where $k = \frac{n-1}{2^t}$

On the other hand, by Fermat's Little Theorem, $(m^k)^{p-1} \equiv 1 \pmod{p}$

So $2^{t+1} \mid p -1$, $p \equiv 1 \pmod{2^{t+1}}$

By properties of congruencies, we get $n \equiv 1 \ (mod\ 2^{t+1})$

Since t is the largest possible number for $2^t \mid n -1$, it is a contradiction, so $n \nmid m^{n-1} +1$

Question 49

When $p_1 \mid m$ or $p_2 \mid m$, we have $p_1p_2 \nmid m^{p_1-p_2} + 1$

Let $(p_1, m) = (p_2, m) = 1$,

If $p_1p_2 \mid m^{p_1-p_2} + 1$, then $p_1p_2m^{p_2} \mid m^{p_1} + m^{p_2} \Rightarrow p_1p_2 \mid m^{p_1} + m^{p_2}$

Hence $m^{p_2} \equiv -m^{p_1} \equiv -m \pmod{p_1}$

Since $(p_1, m) = 1$, so $m^{p_1-p_2-1} \equiv -1 \pmod{p_1}$

Also, we have $m^{p_1-p_2-1} \equiv -1 \pmod{p_2}$

Thus, $m^{p_1-p_2-1} \equiv -1 \pmod{p_1p_2}$

That is, $p_1p_2 \mid m^{p_1-p_2} + 1$, a contradiction [Refer to Q48]

Question 50

Since $2002 \equiv 4 \pmod 9$, $4^3 \equiv 1 \pmod 9$,

$2002 = 667 \times 3 + 1$ so that $2002^{2002} = 4^{2002} = 4 \pmod 9$

Also, $x^3 \equiv \pm 1, 0 \pmod 9$, where x is integer

Thus, $x_1^3, x_1^3 + x_2^3, x_1^3 + x_2^3 + x_3^3 \not\equiv 4 \pmod 9$

Since 2002 = 103 +103 +1 +1, so we get

$2002^{2002} = 2002\,(2002^{667\times 3})$

$= (10 \times 2002^{667})^3 + (10 \times 2002^{667})^3 + (2002^{667})^3 + (2002^{667})^3$

Thus, $n = 4$.

~ End ~